07

In earlier cases, we considered every object as point mass. Even the finite size objects are considered as point mass to understand their motion.

An extended body (any finite size body) is a system of particles. Hence, now we will consider the motion of system. To understand the motion of a system (extended body), we will understand the concept of centre of mass of a system of particles.

SYSTEM OF PARTICLES AND ROTATIONAL MOTION

|TOPIC 1|
Centre of Mass and Rotational Motion

CHAPTER CHECKLIST

- Rigid Body
- Centre of Mass
- Linear Momentum of a System of Particles
- Torque and Angular Momentum
- Equilibrium of a Rigid Body
- Centre of Gravity
- Moment of Inertia
- Rolling Motion

RIGID BODY

Ideally, a body is said to be a rigid body when it has a perfectly definite shape and size. The distance between all pairs of particles of such a body do not change while applying any force on it.

e.g. A wheel can be considered as rigid body by ignoring a little change in its shape.

Kinds of Motion of a Rigid Body

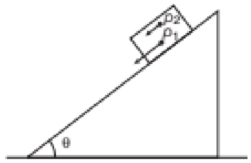
A rigid body can possess pure translational motion, pure rotational motion or a combination of both these motions. Let us explore these kinds of motions one by one.

Pure Translational Motion

This type of motion in which every particle of the body moves through the same linear distance in a straight line and in a given time interval is known as **pure translational motion.**

In pure translational motion, at any instant of time all particles of the body have the same velocity.

In pure translational motion, all the particles of body are moving along parallel paths.

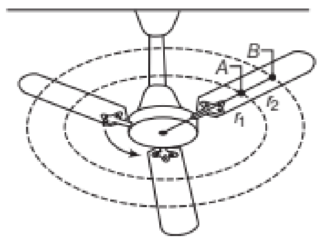


Translational motion of a block down an inclined plane

Rotational Motion (Fixed Axis of Rotation)

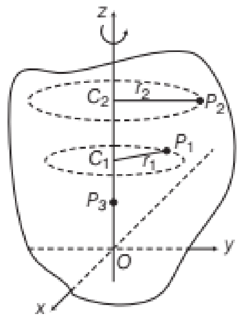
In pure rotational motion, every particle of the rigid body moves in circles of different radii about a fixed line, which is known as **axis of rotation**. e.g.

(i) Consider an oscillating fan as shown in figure. When the fan rotates, the particle A revolves in a circle of radius r_1 and particle B revolves in a circle of radius r_2. Hence, the axis moves side wise, and these points are fixed.



A rotating ceiling fan with the particles A and B moving in circular paths

(ii) The following figure shows the rotational motion of a rigid body about a fixed axis, i.e. z-axis.



A rigid body rotating about the z-axis

The body is in xy-plane. Let P_1 be a particle of the rigid body arbitrarily chosen at a distance r_1 from the fixed axis, i.e. z-axis.

The particle P_1 describes a circle of radius r_1 with its centre C_1 on the fixed axis and the circle lies in a plane, perpendicular to the axis.

The another particle P_2 of the body describes a circle of radius r_2 with centre at C_2 lying on the same z-axis and lies in a plane perpendicular to the axis of rotation.

The circles described by P_1 and P_2 may lie in different planes, both these planes are perpendicular to the fixed axis. For any particle on the axis of rotation like $P_3, r = 0$. Therefore, such a particle remains stationary, while the body rotates.

Some of the examples of pure rotation about an axis are given as,

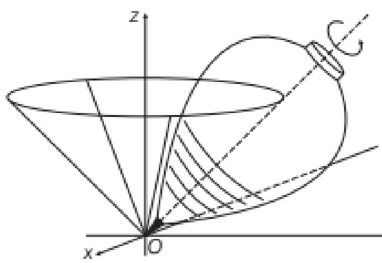
A potter's wheel, a giant wheel in a circus, a merry-go-round etc.

Note

- In pure rotational motion, the plane of the circle in which any particle moves, is always perpendicular to the fixed axis of rotation and has its centre on the axis.
- The particle on the axis of rotation remains stationary.

Precession

In **precession**, one end of axis of rotation is fixed and other end rotates about a circular path.



A spinning top is in precession

(i) The point of contact of the top with ground is fixed. The axis of rotation of the top at any instant passes through the point of contact.

(ii) We mostly deal with the case of fixed axis, so if not stated rotation will be about the fixed axis.

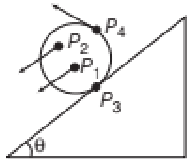
(iii) Oscillating table fan is also an example of precession.

Combination of Translational and Rotational Motions

A rigid body may have a rolling motion, which is a combination of rotation and translation.

e.g. Consider a cylinder rolls down an inclined plane, its motion is the combination of rotation about a fixed axis and translation.

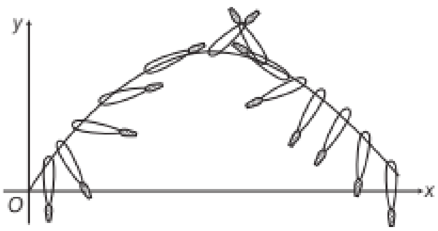
When the cylinder shifts from top to bottom, the points P_1, P_2, P_3, P_4 on this cylinder have different velocities at a particular instant of time. But if the cylinder were to roll without slipping, the velocity of point of contact P_3 would be zero at any instant of time.



A cylinder rolling down

In rolling motion, body is rotating along a fixed line, an axis of rotation and that axis of rotation is in translatory motion. Hence, the motion of a rigid body in some way is either a pure translation or a combination of translation and rotation.

In general, motion of a rigid body is a combination of translation and rotation. As in case of our base ball bat.



Base ball bat neither in pure translation nor in pure rotation but in the combination

Note

The motion of a rigid body which is not pivoted or fixed in some way is either pure translational or a combination of translational and rotational. The motion of a rigid body which is pivoted or fixed in some way is pure rotation.

CENTRE OF MASS

The centre of mass of a body or a system of bodies is the point which moves as though all of the mass were concentrated there and all external forces were applied to it.

Hence, a point at which the entire mass of the body or system of bodies is supposed to be concentrated is known as the **centre of mass**.

For a System of Two Particles

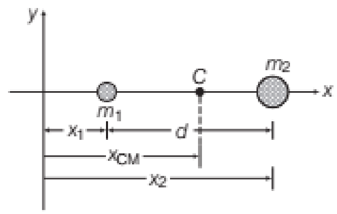
Let the distances of the two particles be x_1 and x_2 respectively from origin O. Let m_1 and m_2 be their masses. The centre of mass of the system is that point C which is at a distance x_{CM} from O, where x_{CM} is given by

$$x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

$$\boxed{\text{Centre of mass, } x_{CM} = \frac{m_1 x_1 + m_2 x_2}{M}}$$

where, $M = m_1 + m_2 =$ total mass of system.

Hence, the centre of mass of a two particles system is such that the product of total mass of the system and the position vector of centre of mass is equal to sum of the products of masses of the two particles and their respective position vectors.



System of two particles

x_{CM} can be regarded as the mass-weighted mean of x_1 and x_2. If the two particles have the same mass $m_1 = m_2 = m$, then

$$x_{CM} = \frac{mx_1 + mx_2}{2m} = \frac{x_1 + x_2}{2}$$

Thus, for two particles of equal mass, the centre of mass lies exactly midway between them.

EXAMPLE |1| A Two Body System

Two bodies of masses 1 kg and 2 kg are located at (1, 2) and (−1, 3), respectively. Calculate the coordinates of the centre of mass.

Sol. Given, $m_1 = 1\,\text{kg}, \quad m_2 = 2\,\text{kg}$

$$x_1 = 1\,\text{m}, \; x_2 = -1\,\text{m}$$

$$y_1 = 2\,\text{m}, \; y_2 = 3\,\text{m}$$

$$\therefore \quad x_{CM} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{1 \times 1 + 2 \times -1}{1 + 2}$$

$$= \frac{1 - 2}{3} = \frac{-1}{3} = -0.33$$

and $$y_{CM} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \frac{1 \times 2 + 2 \times 3}{1 + 2}$$

$$= \frac{2 + 6}{3} = \frac{8}{3} = 2.66$$

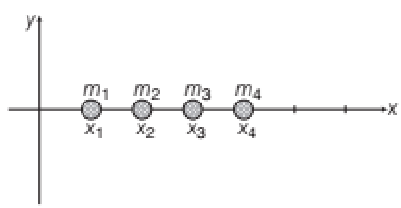
Thus, the coordinates of centre of mass are $(-0.33, \; 2.66)$.

Note

The centre of mass of a system of two particles lies on the straight line joining the two particles.

For a System of *n* Particles

Suppose a system of n particles having masses $m_1, m_2, m_3, \ldots, m_n$ occupying x-coordinates $x_1, x_2, x_3, \ldots, x_n$.



System of *n* particles

i.e. x_{CM} = x-coordinates of centre of mass of system

$$= \frac{m_1x_1 + m_2x_2 + m_3x_3 + \cdots + m_nx_n}{m_1 + m_2 + m_3 + \cdots + m_n}$$

$$\boxed{\text{Centre of mass, } x_{CM} = \frac{\sum_{i=1}^{n} m_i x_i}{\Sigma m_i}}$$

Consider three particles of masses m_1, m_2 and m_3 are lying at points (x_1, y_1), (x_2, y_2) and (x_3, y_3), respectively. Then, the centre of mass of the system of these three particles lies at a point whose coordinates (X, Y) are given by

$$X = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

and $$Y = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

For the particles of equal mass $m = m_1 = m_2 = m_3$, then

$$X = \frac{mx_1 + mx_2 + mx_3}{m + m + m} = \frac{x_1 + x_2 + x_3}{3}$$

and $$Y = \frac{my_1 + my_2 + my_3}{m + m + m} = \frac{y_1 + y_2 + y_3}{3}$$

If particles are distributed in three-dimensional space, then the centre of mass has 3-coordinates, which are

$$x_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i$$

and $$z_{CM} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

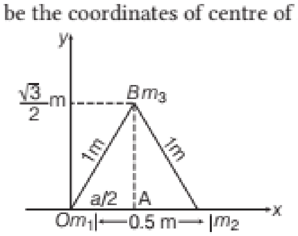
where, $M = m_1 + m_2 + \cdots = \sum_{i=1}^{n} m_i$ is the total mass of the system. The index i runs from 1 to n, m_i is the mass of the ith particle and the position of the ith particle is given by (x_i, y_i, z_i).

EXAMPLE |2| An Equilateral Triangle

Three masses 3 kg, 4 kg and 5 kg are located at the corners of an equilateral triangle of side 1m, then what are the coordinates of centre of mass of this system.

Sol. Suppose the equilateral triangle lies in the xy-plane with mass 3 kg at the origin.

Let (x, y) be the coordinates of centre of mass.



Clearly, $AB = \sqrt{(OB)^2 - (OA)^2} = \sqrt{(1)^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}\text{m}$

Now, $x_1 = 0,\ x_2 = 1\text{m},\ x_3 = OA = 0.5\text{m}$

$m_1 = 3\text{ kg},\ m_2 = 4\text{ kg},\ m_3 = 5\text{ kg}$

$$\therefore \quad x = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}$$

$$= \frac{3 \times 0 + 4 \times 1 + 5 \times 0.5}{3 + 4 + 5}$$

$$= \frac{6.5}{12} = 0.54\text{ m}$$

Again, $y_1 = 0,\ y_2 = 0,\ y_3 = AB = \frac{\sqrt{3}}{2}$,

$$\therefore \quad y = \frac{m_1y_1 + m_2y_2 + m_3y_3}{m_1 + m_2 + m_3}$$

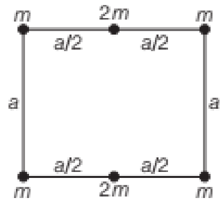
$$= \frac{3 \times 0 + 4 \times 0 + 5 \times \left(\frac{\sqrt{3}}{2}\right)}{3 + 4 + 5}$$

$$= \frac{5 \times \sqrt{3}}{2 \times 12} = 0.36\text{ m}$$

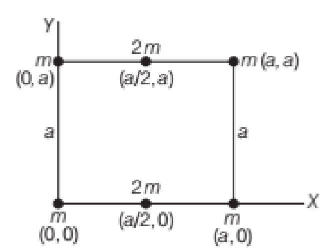
Thus, the coordinate of centre of mass are (0.54 m, 0.36 m).

EXAMPLE |3| Centre of Mass of Six Particles

Six particles are placed at different points of a square as shown in the figure. Find the centre of mass for the system of six particles.



Sol. First of all consider a coordinate system for the given system of particles.



Now, apply the formula for the X-component of centre of mass.

$$X_{CM} = \frac{2m \times a/2 + 2m \times a/2 + m \times a + m \times a}{m + 2m + m + m + 2m + m}$$

$$= \frac{4ma}{8m} = \frac{a}{2}$$

Now, apply the formula for the Y-component of centre of mass.

$$Y_{CM} = \frac{m \times a + 2m \times a + m \times a}{8m} = \frac{4ma}{8m} = \frac{a}{2}$$

So, the coordinates of centre of mass should be written as

$$(X_{CM}, Y_{CM}) = \left(\frac{a}{2}, \frac{a}{2}\right)$$

Relation between Position Vectors of Particles and Centre of Mass

We can also define centre of mass in terms of vector. Let $\mathbf{r}_i$ be the position vector of the ith particle and $\mathbf{R}$ be the position vector of the centre of mass, then

$$\mathbf{r}_i = x_i\hat{\mathbf{i}} + y_i\hat{\mathbf{j}} + z_i\hat{\mathbf{k}}$$

and

$$\mathbf{R} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$$

Position vector of the centre of mass, $\mathbf{R} = \dfrac{\sum_{i=1}^{n} m_i \mathbf{r}_i}{m}$

If the origin of the frame of reference is considered at the centre of mass, then

$$\mathbf{R}_{CM} = 0$$

So,

$$\sum_{i=1}^{n} m_i \mathbf{r}_i = 0$$

Centre of Mass of Rigid Continuous Bodies

For a real body which is a continuous distribution of matter, point masses are then differential mass elements dm and centre of mass is defined as

$$x_{CM} = \frac{1}{M}\int x\, dm, \quad y_{CM} = \frac{1}{M}\int y\, dm$$

$$z_{CM} = \frac{1}{M}\int z\, dm$$

where, M is total mass of that real body.

If we choose the origin of coordinates axes at centre of mass then,

$$\int x\, dm = \int y dm = \int z dm = 0$$

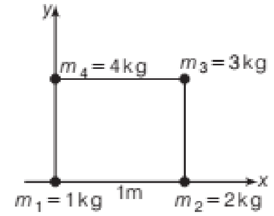
Position of the Centre of Mass in the Absence of External Force

In the absence of an external force, the position of the centre of mass of a stationary system does not change. However, if under the influence of external forces, the position of different particles of masses $m_1, m_2, m_3, \ldots$ changes by $\Delta \mathbf{r}_1, \Delta \mathbf{r}_2, \ldots$ then shift in the position vector of the centre of mass is given by

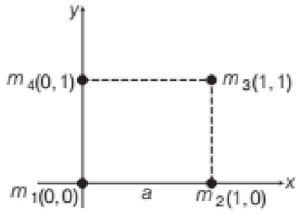
$$\Delta \mathbf{r}_{CM} = \frac{m_1 \Delta \mathbf{r}_1 + m_2 \Delta \mathbf{r}_2 + \ldots + m_n \Delta \mathbf{r}_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum_{i=1}^{n} m_i \Delta \mathbf{r}_i}{\sum_{i=1}^{n} m_i}$$

EXAMPLE |4| Particles at the Corner of a Square

If four particles of mass 1kg, 2kg, 3kg and 4kg are placed at the four vertices A, B, C and D of square of side 1m. Find the position of the centre of mass of the particle.



Sol. Observing the figure,



Coordinates of m_1, $x_1 = 0$, $y_1 = 0$

Coordinates of m_2, $x_2 = 1$, $y_2 = 0$

Coordinates of m_3, $x_3 = 1$, $y_3 = 1$

Coordinates of m_4, $x_4 = 0$, $y_4 = 1$

$$\therefore \quad x_{CM} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

$$= \frac{(1)(0) + (2)(1) + (3)(1) + (4)(0)}{1 + 2 + 3 + 4}$$

$$= \frac{5}{10} = 0.5 \text{ m}$$

Now, $$y_{CM} = \frac{m_1y_1 + m_2y_2 + m_3y_3 + m_4y_4}{m_1 + m_2 + m_3 + m_4}$$

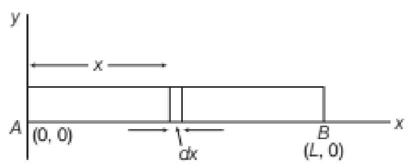
$$= \frac{(1)(0) + (2)(0) + (3)(1) + (4)(1)}{1 + 2 + 3 + 4}$$

$$= \frac{7}{10} = 0.7 \text{ m}$$

Thus, centre of mass is located at (0.5m, 0.7m).

Centre of Mass of a Uniform Thin Rod

Let us consider a uniform thin rod AB of mass M and length L. The rod is held along x-axis with its end A at the origin, where $x = y = 0$, as shown in figure. Suppose the element is placed side by side of length dx at distance i.e. x from the origin.



Calculation of centre of mass of a uniform thin rod

Then, mass of element $dm = \frac{M}{L} dx$

Let (x_{CM}, y_{CM}) be the coordinates of centre of mass of thin rod from origin. Then,

$$x_{CM} = \frac{1}{M}\int_0^L x\, dm = \frac{1}{M}\int_0^L x\left(\frac{M}{L}\right) dx = \frac{1}{L}\frac{x^2}{2}\Big|_0^L = \frac{L}{2}$$

Centre of mass, $x_{CM} = \frac{L}{2}$

and $$y_{CM} = \frac{1}{M}\int y\, dm = 0 \quad [\because y = 0 \text{ for thin rod}]$$

$$y_{CM} = 0$$

Thus, it means that the centre of mass of thin rod AB lies at point $\left(\frac{L}{2}, 0\right)$ i.e. centre between its ends A and B. Thus, the centre of mass coincides with geometric centre.

MOTION OF CENTRE OF MASS

As already defined for a system of n particles, we can relate position vector of centre of mass and that of individual masses as

$$M \cdot \mathbf{r}_{CM} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2 + m_3\mathbf{r}_3 + \cdots + m_n\mathbf{r}_n$$

Differentiating with respect to time, we get

$$M \cdot \frac{d}{dt}\mathbf{r}_{CM} = m_1\frac{d}{dt}\mathbf{r}_1 + m_2\frac{d}{dt}\mathbf{r}_2 + \cdots + m_n\frac{d}{dt}\mathbf{r}_n$$

As masses are constants and only position vectors are variables

$$\Rightarrow M \cdot \mathbf{v}_{CM} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \cdots + m_n\mathbf{v}_n$$

So, $$\boxed{\text{Velocity about centre of mass, } \mathbf{v}_{CM} = \frac{\sum_{i=1}^{n} m_i\mathbf{v}_i}{M}}$$

where, $\mathbf{v} = \frac{d\mathbf{r}}{dt}$, rate of change of position vector is velocity.

Again differentiating the result, we get

$$M\frac{d}{dt}\mathbf{v}_{CM} = m_1\frac{d}{dt}\mathbf{v}_1 + m_2\frac{d}{dt}\mathbf{v}_2 + m_3\frac{d}{dt}\mathbf{v}_3 + \cdots + m_n\frac{d}{dt}\mathbf{v}_n$$

As rate of change of velocity is acceleration, we write

$$a = \frac{dv}{dt}$$

$$M\mathbf{a}_{CM} = m_1\mathbf{a}_1 + m_2\mathbf{a}_2 + \cdots + m_n\mathbf{a}_n$$

So, $$\boxed{\text{Acceleration about centre of mass, } \mathbf{a}_{CM} = \frac{\sum_{i=1}^{n} m_i\mathbf{a}_i}{M}}$$

But, $m_i\mathbf{a}_i$ is the resultant force on the ith particle, so

$$M\mathbf{a}_{CM} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n$$

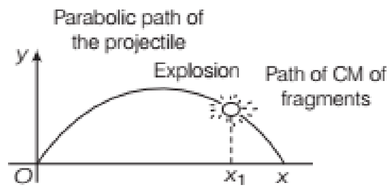
or $$M\mathbf{a}_{CM} = \mathbf{F}_{\text{net, ext}} \quad \ldots\text{(i)}$$

where, $\mathbf{F}_{\text{net, ext}} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots + \mathbf{F}_n$ is net force of all external forces that act on the system.

From the expression (i), it is clear that the centre of mass of a system of particles moves as if whole mass of the system was concentrated at the centre of mass and all the external forces were applied at that point.

Internal forces are action-reaction pairs and hence, cancels out and only vector sum of external forces remains.

e.g. If a rocket explodes in its path and fragment flies in all directions.



Projectile motion of a rocket

As explosion forces are internal to the system, they did not contribute to motion of centre of mass and centre of mass of rocket follows same parabolic trajectory that rocket would have followed if it **would not be exploded.**

So, motion of centre of mass of a system or object is not affected by any of internal forces or they occur always on action-reaction pairs and so their net contribution to acceleration of centre of mass is zero.

Note

The translational motion of extended bodies could be understand with the help of the motion of centre of mass of the system, by considering that the total mass of the system is concentrated on the centre of mass and as all the external forces are acting at the centre of mass.

LINEAR MOMENTUM OF A SYSTEM OF PARTICLES

For a system of n particles, total linear momentum is vector sum of linear momenta of individual particles. Where linear momentum of an individual particle is product of its mass and velocity ($\mathbf{p} = m\mathbf{v}$).

So, linear momentum of system is

$$\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \ldots + \mathbf{p}_n$$

or $$\mathbf{p} = m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \cdots + m_n\mathbf{v}_n \quad \ldots\text{(i)}$$

From the concept of centre of mass, we know that

$$m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + m_3\mathbf{v}_3 + \cdots + m_n\mathbf{v}_n = M \cdot \mathbf{V}_{CM} \quad \ldots\text{(ii)}$$

From Eqs. (i) and (ii), we get

$$\boxed{\text{Total linear momentum, } \mathbf{p} = M\mathbf{V}_{CM}} \quad \ldots\text{(iii)}$$

Thus, the total momentum of a system of particles is equal to the product of the total mass and velocity of its centre of mass.

Newton's Second Law for System of Particles

Now, differentiating Eq. (iii) with respect to time, we get

$$\frac{d}{dt}\mathbf{p} = \frac{d}{dt}M\,\mathbf{V}_{CM}$$

or $$\frac{d\mathbf{p}}{dt} = M\frac{d}{dt}\mathbf{V}_{CM} \quad [\text{if } M \text{ is constant}]$$

or $$\frac{d\mathbf{p}}{dt} = M\,\mathbf{A}_{CM} \quad \ldots\text{(iv)}$$

From centre of mass concept, we know

$$M\mathbf{A}_{CM} = \text{net external force} \quad \ldots\text{(v)}$$

From Eqs. (iv) and (v), we get

$$\therefore \quad \boxed{\text{Net external force, } \mathbf{F}_{ext} = \frac{d\mathbf{p}}{dt}} \quad \ldots\text{(vi)}$$

Above expression represents Newton's second law for system of particles. Here, **p** is net momentum of the system and $\mathbf{F}_{ext}$ is the net external force applied on the system.

Conservation of Linear Momentum for System of particles

If external force is not acting on the system of particles,

$$\mathbf{F}_{ext} = 0, \text{ then } \frac{d\mathbf{p}}{dt} = 0$$

or **rate of change of momentum is zero**, i.e. **momentum of system remains constant.**

So, $$\mathbf{p}_{initial} = \mathbf{p}_{final}, \text{ if } \mathbf{F}_{ext} = 0 \quad \ldots\text{(vii)}$$

Above expression represents the law of conservation of linear momentum for system of particles.

If mass of the system is constant, then

$$M\mathbf{V}_{CM} = \text{constant} \quad [\because \mathbf{F}_{ext} = 0]$$

$$\mathbf{V}_{CM} = \text{constant} \quad \ldots\text{(viii)}$$

Eq. (viii) shows that if the mass of system of particles is constant and if **net external force on the system is zero, then the velocity of centre of mass of the system will remain constant.**

The vector equation $\frac{d\mathbf{p}}{dt} = 0$ or p = constant is equivalent to three equations, $\mathbf{p}_x = C_1$, $\mathbf{p}_y = C_2$ and $\mathbf{p}_z = C_3$, where $\mathbf{p}_x$, $\mathbf{p}_y$ and $\mathbf{p}_z$ are the components of the total linear momentum vector **p** along x, y and z-axes respectively C_1, C_2 and C_3 are constants.

EXAMPLE |5| Enjoying the Ride of a Trolley

A child sits stationary at one end of a long trolley moving uniformly with a speed v on smooth horizontal floor. If child gets up and runs about on the trolley in any manner, what is the speed of CM of the (trolley + child) system? **[NCERT]**

Sol. When the child gets up and runs about on the trolley, he is exerting an internal force on the trolley-child system.

Thus, no external force acts on the system. So, centre of mass is not accelerated. Hence, centre of mass of trolley-child system remains same as before, i.e. v.

EXAMPLE |6| Acceleration of the Position Vectors

If two particles of masses $m_1 = 2$ kg and $m_2 = 3$ kg have position vectors $\mathbf{r}_1 = t\hat{i} + 2t^2\hat{j} + 2t\hat{k}$ and $\mathbf{r}_2 = \hat{\mathbf{i}} + 2t^3\hat{\mathbf{j}} + 5\hat{\mathbf{k}}$ respectively, then the position vector of r_1 and r_2 is in metres and time is in seconds. Calculate the velocity and acceleration of centre of mass of two particles system.

Sol. Here, $m_1 = 2$ kg, $m_2 = 3$ kg and position vectors

$$\mathbf{r}_1 = t\hat{i} + 2t^2\hat{j} + 2t\hat{k}$$

and $$\mathbf{r}_2 = \hat{i} + 2t^3\hat{j} + 5\hat{k}$$

Now, velocity of first particle, $\mathbf{v}_1 = \frac{d}{dt}(\mathbf{r}_1)$

$$= \frac{d}{dt}(t\hat{i} + 2t^2\hat{j} + 2t\hat{k})$$

$$= \hat{i} + 4t\hat{j} + 2\hat{k}$$

Velocity of second particle,

$$\mathbf{v}_2 = \frac{d}{dt}(\mathbf{r}_2)$$

$$= \frac{d}{dt}(\hat{i} + 2t^3\hat{j} + 5\hat{k}) = 6t^2\hat{j}$$

$\therefore$ Velocity of centre of mass of two particles system is

$$\mathbf{v}_{CM} = \frac{m_1\mathbf{v}_1 + m_2\mathbf{v}_2}{m_1 + m_2}$$

$$= \frac{2 \times (\hat{i} + 4t\hat{j} + 2\hat{k}) + 3 \times (6t^2\hat{j})}{2+3}$$

$$= \frac{2\hat{i} + (8t + 18t^2)\hat{j} + 4\hat{k}}{5}$$

$$= \left[\frac{2}{5}\hat{i} + \frac{(8t + 18t^2)}{5}\hat{j} + \frac{4}{5}\hat{k}\right] \text{m/s}$$

Now, acceleration of first particle,

$$\mathbf{a}_1 = \frac{d}{dt}(\mathbf{v}_1) = \frac{d}{dt}(\hat{i} + 4t\hat{j} + 2\hat{k}) = 4\hat{j}$$

Acceleration of second particle

$$\mathbf{a}_2 = \frac{d}{dt}(\mathbf{v}_2) = \frac{d}{dt}(6\,t^2\hat{\mathbf{j}}) = 12\,t\,\hat{\mathbf{j}}$$

$\therefore$ Acceleration of centre of mass of two particles system is calculated as

$$\mathbf{a}_{CM} = \frac{m_1\mathbf{a}_1 + m_2\mathbf{a}_2}{m_1 + m_2}$$

$$= \frac{2(4\,\hat{\mathbf{j}}) + 3(12\,t\,\hat{\mathbf{j}})}{2+3}$$

$$\mathbf{a}_{CM} = \left[\frac{(8+36\,t)}{5}\,\hat{\mathbf{j}}\right]\ \text{m/s}^2$$

TORQUE AND ANGULAR MOMENTUM

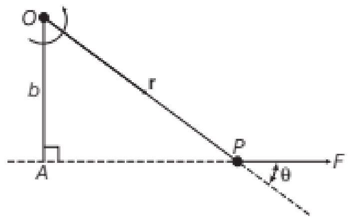
The two physical quantities torque and angular momentum are to be discussed as vector products of two vectors.

Moment of Force (Torque)

Torque is also known as **moment of force** or **couple**. When a force acts on a particle, the particle does not merely move in the direction of the force but it also turns about some point.

So, we can define the torque for a particle about a point as the vector product of position vector of the point where the force acts and with the force itself. Let us consider a particle P and force $\mathbf{F}$ acting on it.

Let the position vector of point of application of force about O is $\mathbf{r}$.



Particle showing torque, position vector and force

From figure, the torque τ acting on the particle relative to the fixed point O is a vector quantity and is defined as

$$\boxed{\text{Torque, } \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}}$$

The magnitude of torque $|\tau|$ is

$$\tau = rF\sin\theta$$

$$= Fr\sin\theta$$

$$\tau = Fr_{\perp} = Fb \text{ (from figure)}$$

Here, $r_{\perp}$ is the perpendicular distance of the line of action of $\mathbf{F}$ from the point O.

Magnitude of torque can also be described as

$$\tau = rF\sin\theta$$

$$\tau = rF_{\perp}$$

Here, $F_{\perp}$ is the component of $\mathbf{F}$ in the direction perpendicular to $\mathbf{r}$.

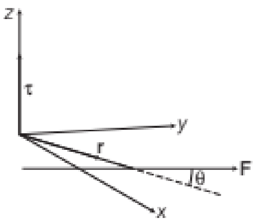
Note

Torque could be zero ($\tau = 0$), if

$r = 0$ or $F = 0$ or $\theta = 0, 180°$

In SI system, torque is expressed in Newton-metre (N-m), it has same dimensions as that of workdone. Dimensions of torque are $[ML^2T^{-2}]$.

According to vector product, torque is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{F}$ as shown below.



Plane showing torque perpendicular to force

If the body is rotating counterclockwise, then the torque is taken positive otherwise negative.

EXAMPLE |7| Finding the Torque

Find the torque of a force $7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$ about the origin. The force acts on a particle whose position vector is $\hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$. **[NCERT]**

Sol. Given, position vector

$$\mathbf{r} = \hat{\mathbf{i}} - \hat{\mathbf{j}} + \hat{\mathbf{k}}$$

Force, $\mathbf{F} = 7\hat{\mathbf{i}} + 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$

Torque, $\tau = ?$

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 1 & -1 & 1 \\ 7 & 3 & -5 \end{vmatrix}$$

$$= (5-3)\hat{\mathbf{i}} - (-5-7)\hat{\mathbf{j}} + (3+7)\hat{\mathbf{k}} = 2\hat{\mathbf{i}} + 12\hat{\mathbf{j}} + 10\hat{\mathbf{k}}$$

Angular Momentum of a Particle

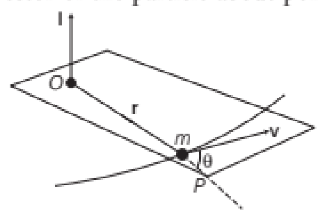
Angular momentum (**L**) can be defined as **moment of linear momentum** about a point.

The angular momentum of a particle of mass m moving with velocity $\mathbf{v}$ (having a linear momentum $\mathbf{p} = m\mathbf{v}$) about a point O is defined by the following vector product.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

or $\boxed{\text{Angular momentum, } \mathbf{L} = m(\mathbf{r} \times \mathbf{v})}$

$\mathbf{L}$ is angular momentum of particle about point O.
$\mathbf{r}$ is position vector of the particle about point O.



A particle moving about a point O

Magnitude of angular momentum depends on the position of point O and is given by $L = mvr \sin\theta$

where, θ = angle between $\mathbf{r}$ and $\mathbf{v}$

L can be represented as $L = rp \sin\theta = rp_{\perp}$

where, $p_{\perp}$ $(p \sin\theta)$ is the component of $\mathbf{p}$ in a direction perpendicular to $\mathbf{r}$.

and $\quad L = pr \sin\theta = pr_{\perp}$

where, $r_{\perp}$ is the perpendicular distance of the linear momentum vector $(\mathbf{p})$ from origin.

EXAMPLE |8| Revolving Electron in an Atom

In a hydrogen atom, electron revolves in a circular orbit of radius 0.53 Å with a velocity of 2.2×10^6 m/s with an angle 30°. If the mass of electron is 9×10^{-31} kg. Find its angular momentum.

Sol. Given, $r = 0.53 \text{Å} = 0.53 \times 10^{-10}$ m

$m = 9 \times 10^{-31}$ kg, $v = 2.2 \times 10^6$ m/s,

$\theta = 30°$, $L = ?$

$$L = mvr \sin\theta$$
$$= 9 \times 10^{-31} \times 2.2 \times 10^6 \times 0.53 \times 10^{-10} \times \sin 30°$$
$$= 5.247 \times 10^{-3} \text{ kg-m}^2/\text{s}$$

Relation between Torque and Angular Momentum

As angular momentum,

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \text{ or } \mathbf{L} = m(\mathbf{r} \times \mathbf{v})$$

Differentiating with respect to time,

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} m(\mathbf{r} \times \mathbf{v})$$

or $$\frac{d\mathbf{L}}{dt} = m\left(\frac{d\mathbf{r}}{dt} \times \mathbf{v} + \mathbf{r} \times \frac{d\mathbf{v}}{dt}\right)$$

[from product rule of differentiation]

But $$\frac{d\mathbf{r}}{dt} = \mathbf{v} \text{ and } \mathbf{v} \times \mathbf{v} = 0$$

So, $$\frac{d\mathbf{L}}{dt} = m\left(\mathbf{r} \times \frac{d\mathbf{v}}{dt}\right)$$

But $$\frac{dv}{dt} = a = \text{acceleration}$$

$$= m(\mathbf{r} \times \mathbf{a}) = (\mathbf{r} \times m\mathbf{a}) = \mathbf{r} \times \mathbf{F}$$

As $$\tau = \mathbf{r} \times \mathbf{F}$$

$\therefore$ Rate of change of angular momentum, $\dfrac{d\mathbf{L}}{dt} = \tau$

Above equation gives Newton's second law of motion in angular form, **rate of change of angular momentum is equal to the torque applied.**

Torque and Angular Momentum for a System of Particles

For a system of particles or a rigid body with n-particles, the total angular momentum of the system is the vector sum of the angular momenta of the individual particles.

$$\mathbf{L} = \mathbf{L}_1 + \mathbf{L}_2 + \mathbf{L}_3 + \cdots + \mathbf{L}_n = \sum_{i=1}^{n} \mathbf{L}_i$$

Differentiating the above expression with time,

$$\frac{d\mathbf{L}}{dt} = \sum_{i=1}^{n} \frac{d\mathbf{L}_i}{dt}$$

But $\sum_{i=1}^{n} \dfrac{d\mathbf{L}_i}{dt}$ is the net torque on the system so, $\dfrac{d\mathbf{L}}{dt} = \tau_{net}$

Above expression shows Newton's second law for system of particles which state that the net external torque τ_{net} acting on a system of particles is equal to the time rate of change of the system's total angular momentum $\mathbf{L}$.

For ith particle, the torque is given by

$$\tau = \mathbf{r}_i \times \mathbf{F}_i$$

The force $\mathbf{F}_i$ on the ith particle is the vector sum of external forces $\mathbf{F}_i^{ext}$ acting on the particle and the internal forces

total torque is given by

$$\tau = \sum_i \tau_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i = \tau_{ext} + \tau_{int}$$

where, $\quad \tau_{ext} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext}$ and $\tau_{int} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{int}$

According to Newton's third law of motion, $\tau_{int} = 0$, then $\tau_{ext} = \tau$.

$$\therefore \quad \tau = \sum \tau_i$$

Rate of change of angular momentum, $\dfrac{d\mathbf{L}}{dt} = \tau_{ext}$

Thus, the time rate of the total angular momentum of a system of particles about a point is equal to the sum of the external torques acting on the system taken about the same point.

Relation between Angular Momentum and Moment of Inertia

For a rigid body (about an fixed axis),

L = sum of angular momenta of all particles

$= m_1 v_1 r_1 + m_2 v_2 r_2 + \ldots\ldots$

$= m_1 r_1^2 \omega + m_2 r_2^2 \omega + \ldots\ldots$ $[\because v = \omega r]$

$= (m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \ldots\ldots)\omega$

i.e. $L = I\omega$

Conservation of Angular Momentum

As, $\frac{d\mathbf{L}}{dt} = \tau_{net}$ for the rigid body which can be treated as a system of n-particles. If $\tau_{ext} = 0$, then $\frac{d\mathbf{L}}{dt} = 0$, that means

$\mathbf{L}$ = constant with time.

Hence, **without any external torque, angular momentum of a system of particles remain constant.** This is known as the conservation of angular momentum.

If $\tau_{ext} = 0 \Rightarrow \boxed{\mathbf{L}_{initial} = \mathbf{L}_{final}}$

If $\mathbf{L}$ is constant, then its all three components will also be constant.

L_x = constant, L_y = constant, L_z = constant

Applications of Law of Conservation of Angular Momentum

- A circus acrobat performs beats involving spin by bringing her arms and legs closer to her body or *vice-versa*. On bringing the arms and legs closer to the body, her moment of inertia I decreases, hence angular velocity ω increases. The same principle is applied by ice skater or a ballet dancer.
- All helicopters are provided with two propellers. If there were only one propeller, the helicopter would rotate itself in opposite direction.
- A diver performs somersaults by jumping from a high diving board keeping his legs and arms out stretched first and then curling his body. On doing so, the moment of inertia I of his body decreases. As angular momentum remains constant, therefore, angular velocity ω of his body increases. He, then, performs somersaults. As the diver is about to touch the surface of water, he stretches out his limbs.

EXAMPLE |9| Angular Momentum of an Object

At a certain time, a 0.25 kg object has a position vector $\mathbf{r} = 2\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$ m. At that instant, its velocity is $\mathbf{v} = -5\hat{\mathbf{i}} + 5\hat{\mathbf{k}}$ m/s and the force acting on it is $\mathbf{F} = 4\hat{\mathbf{j}}$ N.

(i) What is the angular momentum of the object about the origin?

(ii) What is torque on it?

Sol. Angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{p} = \mathbf{r} \times m\mathbf{v} = m(\mathbf{r} \times \mathbf{v})$

Now, $\mathbf{r} \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 0 & -2 \\ -5 & 0 & 5 \end{vmatrix}$

$= \hat{\mathbf{i}}[(0 \times 5) - (0 \times -2)] - \hat{\mathbf{j}}[(2 \times 5) - (-2 \times -5)] + \hat{\mathbf{k}}[2 \times 0 - (0 \times -5)]$

$= 0$

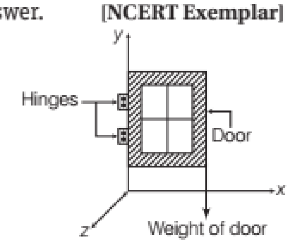
So, angular momentum of particle is zero.

and τ = torque $= \mathbf{r} \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ 2 & 0 & -2 \\ 0 & 4 & 0 \end{vmatrix} = 8\hat{\mathbf{i}} + 8\hat{\mathbf{k}}$ N-m

EXAMPLE |10| A Freely Rotating Body

A door is hinged at one end and is free to rotate about a vertical axis. Does its weight cause any torque about its axis? Give reason for your answer. **[NCERT Exemplar]**

Sol. As torque $= \mathbf{r} \times \mathbf{F}$



That means torque produced by force is in a plane perpendicular to plane containing $\mathbf{r}$ and $\mathbf{F}$.

So, if door is in xy-plane, torque produced by weight is in $\pm z$-direction.

It is never about an axis passing through y-direction.

EQUILIBRIUM OF A RIGID BODY

A rigid body is said to be in equilibrium, if both of its linear momentum and angular momentum are not changing with time.

Thus, equilibrium body does not possess linear acceleration or angular acceleration.

This means,

(i) The total force, i.e. the vector sum of all forces acting on the rigid body is zero.

$$\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \sum_{i=1}^{n} \mathbf{F}_i = 0$$

If the total force acting on the body is zero, then the linear momentum of body remains constant, so body must be in translatory equilibrium.

The above equations can be written in scalar form as

$$\sum_{i=1}^{n} F_{ix} = 0, \sum_{i=0}^{n} F_{iy} = 0 \text{ and } \sum_{i=0}^{n} F_{iz} = 0$$

where, F_{ix}, F_{iy} and F_{iz} are x, y and z-components of force $\mathbf{F}_i$.

(ii) The total torque, i.e. the vector sum of all torques acting on the body must be zero.

$$\tau_1 + \tau_2 + \tau_3 + \cdots + \tau_n = \sum_{i=1}^{n} \tau_i = 0$$

The above equation can be written in scalar form as

$$\sum_{i=0}^{n} \tau_{ix} = 0, \ \sum_{i=0}^{n} \tau_{iy} = 0$$

and $$\sum_{i=0}^{n} \tau_{iz} = 0.$$

where, τ_{ix}, τ_{iy} and τ_{iz} are x, y and z-components of torque τ_i.

Note

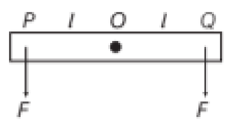
Rotational equilibrium condition remains valid for any origin.

A body may remain in **partial equilibrium** as well. It means that body may remain only in **translational equilibrium** or only in **rotational equilibrium**.

(iii) The sum of the components of the torques along any axis perpendicular to the plane of the forces must be zero.

Rotational Equilibrium Only

If the net torque acting on the rigid body is zero but net force is non-zero, then rigid body is in **rotational equilibrium only**.

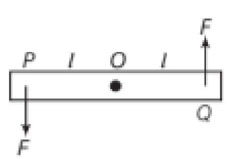


A rod under rotational equilibrium

Two equal forces are acting at the ends of a rod. In this case, the net torque produced by two forces will be zero, but net force is non-zero. It means that rigid body will have zero angular acceleration but non-zero linear acceleration.

Translational Equilibrium Only

If the net force acting on the rigid body is zero but net torque is non-zero, then rigid body is in **translational equilibrium only**.

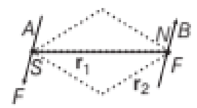


A rod under translational equilibrium

Two equal and opposite forces are acting at the ends of a rod. In this case the net force is zero, but both the forces will produce a non-zero torque. It means that rigid body will have zero linear acceleration but non-zero angular acceleration.

COUPLE

A pair of equal and opposite forces with parallel lines of action are known as a **couple**. A couple produces rotation without translation.



Equal and opposite forces

Let a couple with forces $-\mathbf{F}$ and $\mathbf{F}$ acting at A and B points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ with respect to some origin O.

The moment of couple = total torque

$$= \mathbf{r}_1 \times (-\mathbf{F}) + \mathbf{r}_2 \times \mathbf{F}$$

$$= \mathbf{r}_2 \times \mathbf{F} - \mathbf{r}_1 \times \mathbf{F} = (\mathbf{r}_2 - \mathbf{r}_1) \times \mathbf{F}$$

$\because$ $\boxed{\text{Moment of couple} = \mathbf{AB} \times \mathbf{F}}$

By triangle law, $\mathbf{r}_1 + \mathbf{AB} = \mathbf{r}_2$

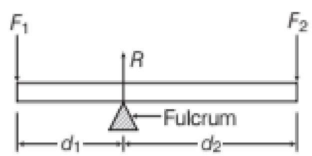
So, $\mathbf{AB} = \mathbf{r}_2 - \mathbf{r}_1$

So, moment of couple is independent of origin.

Principle of Moment

When an object is in rotational equilibrium, then algebraic sum of all torques acting on it is zero. Clockwise torques are taken negative and anti-clockwise torques are taken positive.

e.g. Consider a rod of negligible mass is provided at some point like a see-saw or a lever. Pivot of lever is called **fulcrum**.



A fulcrum point in a rod

For translational equilibrium,

$$\Sigma F = 0 \text{ or } R - F_1 - F_2 = 0$$

For rotational equilibrium, $\Sigma\tau = 0$ or $F_1 d_1 - F_2 d_2 = 0$

In case of levers, F_1 is called load, d_1 is **load arm**, F_2 is called **effort**, d_2 is **effort arm**.

As, $F_1 d_1 - F_2 d_2 = 0$ or $F_1 d_1 = F_2 d_2$

or load × load arm = effort × effort arm.

This is called **principle of moment** for a lever.

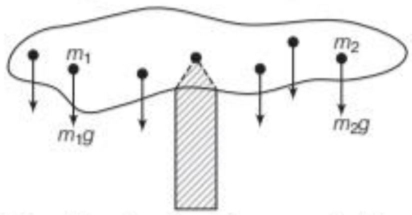
Also, $\boxed{\text{Mechanical advantage, } \dfrac{F_1}{F_2} = \dfrac{d_2}{d_1}}$

The ratio $\frac{F_1}{F_2}$ is called **mechanical advantage** (MA).

Mechanical advantage greater than one is usually required as this means a small effort is required to lift a large load.

CENTRE OF GRAVITY

If a body is supported on a point such that total gravitational torque about this point is zero, then this point is called **centre of gravity** of the body.



Calculation of centre of gravity supported at a point

As, total gravitational torque about centre of gravity (CG) is zero.

Then, $\tau_g = \sum_{i=1}^{n} \tau_i = \Sigma \mathbf{r}_i \times mg = \mathbf{g} \cdot \Sigma m_i \mathbf{r}_i = 0$

[if g = constant]

So, $\Sigma m_i \mathbf{r}_i = 0$ [Ag $\neq 0$] ...(i)

Above equation shows that with centre of gravity as origin,

$$\Sigma m_i \mathbf{r}_i = 0$$

If r is position vector of centre of mass $= \frac{1}{M} \cdot \Sigma m_i \cdot \mathbf{r}_i$

If centre of mass lies at origin

$$0 = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2 + \cdots + m_n \mathbf{r}_n, \ \Sigma m_i \mathbf{r}_i = 0 \quad \ldots\text{(ii)}$$

So, centre of gravity coincides with centre of mass if g is constant. But, above statement is not true for very large objects as g will vary and CG does not coincide with CM.

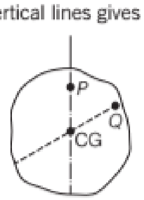
Note

- CM and CG are two different concepts. For small bodies, g will
- be constant at all points on the body.

Centre of Gravity of a Body having Irregular Shape

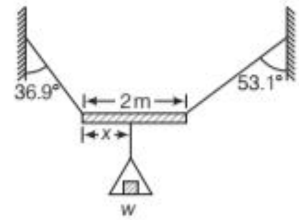
If we suspended the body from a point say P, the vertical line through P passes through centre of gravity. Then, suspend the body through other point Q.

The intersection of vertical lines gives centre of gravity.



EXAMPLE |11| A Non-uniform Rod

A non-uniform bar of weight w is suspended at rest by two strings of negligible weight as shown,



Angles made by strings with vertical are 36.9° and 53.1°, respectively. The bar is 2 m long. Calculate the distance x of the centre of gravity of bar from its left end.

[NCERT]

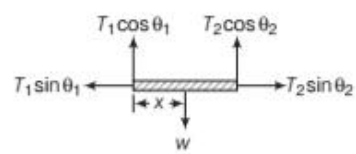
Sol. Bar is in equilibrium, so the total of forces and torque acting on it must be zero.

$\Rightarrow \quad \Sigma F = 0$

and $\quad \Sigma \tau = 0$

Let, T_1 and T_2 are tensions in strings as shown

Then, force components are



As, sum of vertical components is zero

$\therefore \quad T_1 \cos \theta_1 + T_2 \cos \theta_2 - w = 0 \quad$ ...(i)

Sum of horizontal components is zero,

$\Rightarrow \quad T_1 \sin \theta_1 = T_2 \sin \theta_2 \quad$ ...(ii)

Sum of torques must be zero, (taking torque about left hand corner.)

$$w \cdot x = T_2 \cos \theta_2 \times 4 \quad \ldots\text{(iii)}$$

From Eq. (ii), $T_1 = \frac{T_2 \sin \theta_2}{\sin \theta_1}$

Substituting in Eq. (i), we get

$$T_2 = w \sin \theta_1 / \sin \theta_2 \cos \theta_1 + \cos \theta_2 \sin \theta_1$$

So, substituting for T_2 in Eq. (iii), we get

$$x = \frac{L \cdot \sin\theta_1 \cos \theta_2}{\sin \theta_2 \cos \theta_1 + \cos \theta_2 \sin \theta_1}$$

$$= L \cdot \frac{\sin \theta_1 \cos \theta_2}{\sin (\theta_1 + \theta_2)}$$

Here, $L = 2\text{m}, \theta_1 = 36.9°, \theta_2 = 53.1°$

So, $\theta_1 + \theta_2 = 90°$

So, $x = 2 \times \sin 36.9° \times \cos 53.1°$

$= 2.20$ m

EXAMPLE |12| Analyse the Height of Large Objects

Centre of gravity of a body on the earth coincides with its centre of mass for a small object and for a large object, it may not.

What is qualitative meaning of small and large in this regard? For which of the following two of them coincides, a building, a pond, a lake, a mountain. **[NCERT Exemplar]**

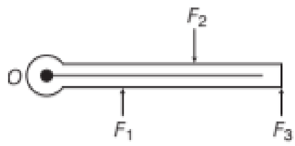
Sol. Centre of mass and centre of gravity are two different concepts. But if g does not vary from one part of body to other than CG and CM coincides.

So, when vertical height of the object is very small compared to radius of earth, we call object small, otherwise we call it extended. In above context, building and pond are small objects and a deep lake and a mountain are large extended objects.

EXAMPLE |13| Non-concurrent Forces

The vector sum of a system of non-collinear forces acting on a rigid body is given to be non-zero. If vector sum of all the torques due to this system of forces about a certain point is found to be zero, does this means that it is necessarily zero about any arbitrary point. **[NCERT Exemplar]**

Sol. Given that $\sum_{i=1}^{n} F_i = 0$



Sum of torques about a point O,

$$\Sigma\, \mathbf{r}_i \times \mathbf{F}_i = 0$$

Then, sum of torques about any other point A is

$$\Sigma\tau = \Sigma(\mathbf{r}_i - \mathbf{a}) \times F_i = \Sigma\, \mathbf{r}_i\mathbf{F}_i - \Sigma\, \mathbf{a} \times \mathbf{F}_i$$

Clearly, second term is non-zero. Hence, sum of torques about any other point may not be zero.

TOPIC PRACTICE 1

OBJECTIVE Type Questions

1. For n particles in a space, the suitable expression for the x-coordinate of the centre of mass of the system is

(a) $\frac{\Sigma m_i x_i}{m_i}$ (b) $\frac{\Sigma m_i x_i}{M}$ (c) $\frac{\Sigma m_i y_i}{M}$ (d) $\frac{\Sigma m_i z_i}{M}$

Here, M is the total mass of the system.

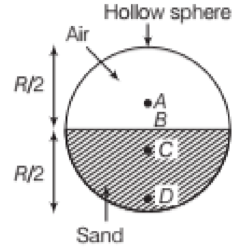
Sol. (b) For system of n- particles in space, the centre of mass of such a system is at (X, Y, Z), where

$$X = \frac{\Sigma m_i x_i}{M}, \quad Y = \frac{\Sigma m_i y_i}{M} \text{ and } Z = \frac{\Sigma m_i z_i}{M}$$

Here, $M = \Sigma m_i$ is the total mass of the system. The index i runs from 1 to n.

$m_i \rightarrow$ Mass of the ith particle and position of ith particle is (x_i, y_i, z_i).

2. Which of the following points is the likely position of the centre of mass of the system shown in figure? **[NCERT Exemplar]**



(a) A (b) B (c) C (d) D

Sol. (c) Centre of mass of a system lies towards the part of the system, having bigger mass. In the above diagram, lower part is heavier, hence CM of the system lies below the horizontal diameter.

3. A body is rotating with angular velocity $\omega = (3\hat{i} - 4\hat{j} - \hat{k})$. The linear velocity of a point having position vector $\mathbf{r} = (5\hat{i} - 6\hat{j} + 6\hat{k})$ is

(a) $6\hat{i} + 2\hat{j} - 3\hat{k}$ (b) $-18\hat{i} - 23\hat{j} + 2\hat{k}$

(c) $-30\hat{i} - 23\hat{j} + 2\hat{k}$ (d) $6\hat{i} - 2\hat{j} + 8\hat{k}$

Sol. (c) Here, $\omega = 3\hat{i} - 4\hat{j} - \hat{k}$, $\mathbf{r} = 5\hat{i} - 6\hat{j} + 6\hat{k}$

$$\text{As, } \mathbf{v} = \omega \times \mathbf{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -4 & -1 \\ 5 & -6 & 6 \end{vmatrix}$$

$$= \hat{i}(-24 - 6) + \hat{j}(-5 - 18) + \hat{k}(-18 + 20)$$

$$= -30\hat{i} - 23\hat{j} + 2\hat{k}$$

4. Newton's second law for rotational motion of a system of particles can be represented as ($\mathbf{L}$ is the angular momentum for a system of particles)

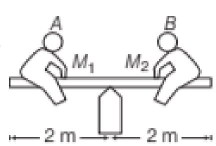
(a) $\frac{d\mathbf{p}}{dt} = \tau_{ext}$ (b) $\frac{d\mathbf{L}}{dt} = \tau_{ext}$

(c) $\frac{d\mathbf{L}}{dt} = \tau_{ext}$ (d) $\frac{d\mathbf{L}}{dt} = \tau_{int} + \tau_{ext}$

Sol. (c) Newton's second law for rotational motion is

$$\tau = \tau_{ext} = \frac{d\mathbf{L}}{dt}$$

5. In the game of see-saw, what should be the displacement of boy B from right edge to keep the see-saw in equilibrium? ($M_1 = 40$ kg, $M_2 = 60$ kg)



(a) $\frac{4}{3}$ m (b) 1 m (c) $\frac{2}{3}$ m (d) Zero

Sol. (c) For the equilibrium, $M_1 g \times r_A = M_2 g \times x$

$$(40 \times 10) \times 2 = (60 \times 10) x$$

$$x = \frac{8}{6} = \frac{4}{3} \text{ m}$$

So, 60 kg boy has to be displaced $= 2 - \frac{4}{3} = \frac{2}{3}$ m

VERY SHORT ANSWER Type Questions

6. Should there necessarily be any mass at centre of mass of system?

Sol. No, the centre of mass of a ring lies at its centre.

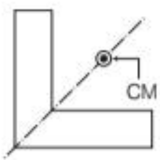
7. Is centre of mass a reality?

Sol. No, the centre of mass of a system is a hypothetical point which acts as a single mass particle of the system for an external force.

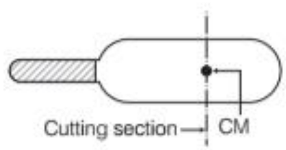
8. In which case centre of mass of a body lie outside it? Give one example.

Sol. If geometrical centre of a body lies outside it, then centre of mass of body lies outside the body.

Centre of mass of a *L*-shaped lamina lies outside it on the line of symmetry.



9. A cricket bat is cut through its centre of mass into two parts as shown



Then, state whether both parts are of same mass or not. Also, give reason.

Sol. Centre of mass of a body lies towards region of heavier mass. So, if bat is cut through its centre of mass, both parts are not of equal masses.

10. $(n-1)$ equal point masses each of mass m are placed at the vertices of a regular n-polygon. The vacant vertex has a position vector **a** with respect to the centre of the polygon. Find the position vector of centre of mass. **[NCERT Exemplar]**

Sol. Suppose, **b** be the position vector of centre of mass of regular n-polygon.

As $(n-1)$ equal point masses each of mass m are placed at $(n-1)$ vertices of regular polygon, therefore

$$\frac{(n-1)mb + ma}{(n-1+1)m} = 0 \Rightarrow (n-1)mb + ma = 0$$

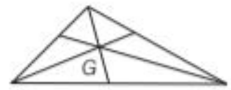
$$\Rightarrow \quad b = \frac{-a}{(n-1)}$$

11. Find centre of mass of a triangular lamina. **[NCERT]**

Sol. For any planar solid, centre of mass always lies at its geometrical centre.

Geometrical centre of a triangle is intersection point of its media.

So, for any given triangular lamina



Centre of mass is at its centroid, point of intersection is media.

12. If no external torque act on a body, will its angular velocity remain conserved?

Sol. No, angular velocity is not conserved but angular momentum is conserved.

13. Which component of linear momentum does not contribute to angular momentum?

Sol. The radial component of linear momentum does not contribute to angular momentum.

14. If net torque on a rigid body is zero, does its linear momentum necessary remain conserved?

Sol. The linear momentum remain conserved if the net force on the system is zero.

15. What happens to the moment of force about a point, if the line of action of the force moves towards the point?

Sol. Moment of force

= force × the perpendicular distance of the line of action of force from the axis of rotation.

Hence, the moment of force about a point decreases if the line of action of the force moves towards that point.

16. The vector sum of a system of non-collinear forces acting on a rigid body is given to be non-zero. If the vector sum of all the torques due to the system of forces about a certain point is found to be zero, does this mean that it is necessarily zero about any arbitrary point? **[NCERT]**

Sol. No, given $\sum_i \mathbf{F}_i \neq 0$

The sum of torques about a certain point O

$$\sum_i \mathbf{r}_i \times \mathbf{F}_i = 0$$

The sum of torques about any other point O'.

$$\sum_i (\mathbf{r}_i - \mathbf{a}) \times \mathbf{F}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i - \mathbf{a} \times \sum_i \mathbf{F}_i$$

Here, the second term need not vanish.

17. When is a body lying in a gravitation field in stable equilibrium?

Sol. A body in a gravitation field will be in stable equilibrium, if the vertical line through its centre of gravity passes through the base of the body.

18. Is centre of mass and centre of gravity body always coincide?

Sol. No, if the body is large such that g varies from one point to another, then centre of gravity is offset from centre of mass.

But for small bodies, centre of mass and centre of gravity lies at their geometrical centres.

19. Does the centre of mass of a solid necessarily lie within the body? If not, give an example.

Sol. No, the centre of mass of L-shaped rod lies in the region outside of rod.

20. A faulty balance with unequal arms has its beam horizontal. Are the weights of the two pans equal?

Sol. They are of unequal mass. Their masses are in the inverse ratio of the arms of the balance.

21. When a labourer cuts down a tree, he makes a cut on the side facing the direction in which he wants it to fall. Why?

Sol. The weight of tree exerts a torque about the point where the cut is made. This causes rotation of the tree about the cut.

22. Why a wrench of longer arm is preferred in comparison to a wrench of shorter arm?

Sol. The torque applied on the nut by the wrench is equal to the force multiplied by the perpendicular distance from the axis of rotation. Hence, to increase torque a wrench of longer arm is preferred.

23. Why in hand driven grinding machine, handle is put near the circumference of the stone or wheel?

Sol. For a given force, torque can be increased if the perpendicular distance of the point of application of the force from the axis of rotation is increased.

Hence, the handle put near the circumference produces maximum torque.

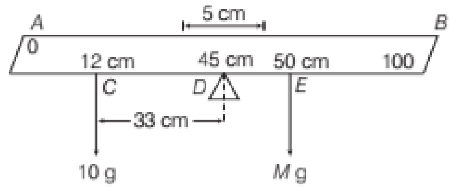
24. The bottom of a ship is made heavy. Why?

Sol. The bottom of a ship is made heavy so that its centre of gravity remains low. This ensures the stability of its equilibrium.

SHORT ANSWER Type Questions

25. A metre stick is balanced on a knife edge at its centre. When two coins, each of mass 5 g are put one on top of the other at the 12 cm mark, the stick is found to be balanced at 45.0 cm. What is the mass of the metre stick ? **[NCERT]**

Sol. Let total mass of the metre stick be M kg.



Distance between mid-point E and new centre of gravity (DE),

$$= 50 - 45 = 5 \text{ cm}$$

Distance between 12 cm mark and new centre of gravity (CD),

$$= 45 - 12 = 33 \text{ cm}$$

From principle of moments in equilibrium,

$$M \times DE = (2 \times 5) \times CD$$

$$M \times 5 = 10 \times 33 \quad \text{or } M = 66 \text{ g}$$

$\therefore$ Mass of the metre stick is 66 g.

26. In HCl molecule, separation of nuclei of the two atoms is about 1.27Å ($1\text{Å} = 10^{-10}$m). Find the centre of mass of molecule given that a chlorine atom is about 35.5 times massive than a hydrogen atom. **[NCERT]**

Sol. Mass of an atom is concentrated at its nucleus, they can be treated as point masses.

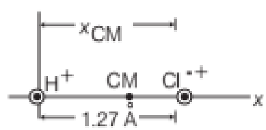
We take nucleus of hydrogen atom at origin and nucleus of chlorine atom on x-axis.

where,

m_1 = mass of hydrogen = 1

m_2 = mass of chlorine = 35.5

$x_1 = 0,\ x_2 = 1.27$ Å



As the system is symmetrical about x-axis, its centre of mass lies on x-axis.

Using $$x_{CM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2} = \frac{1 \times 0 + 35.5 \times 1.27}{1 + 35.5}$$

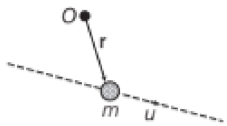
$$= 1.235 \text{ Å}$$

$$= 1.235 \times 10^{-10} \text{m (from origin)}$$

So, centre of mass of HCl molecule is nearly 1.235 Å from the H-nucleus on the line joining H and Cl nuclei.

27. Does angular momentum of a body in translatory motion is zero?

Sol. Angular momentum of a body is measured with respect to certain origin.



So, a body in translatory motion can have angular momentum.

It will be zero, if origin lies on the line of motion of particle.

LONG ANSWER Type I Questions

28. Give the location of the centre of mass of a
(i) sphere
(ii) cylinder
(iii) ring and
(iv) cube each of uniform mass density.

Does the centre of mass of a body necessarily lie inside the body? **[NCERT]**

Sol. (i) Centre of mass of a sphere lies at its geometrical centre.

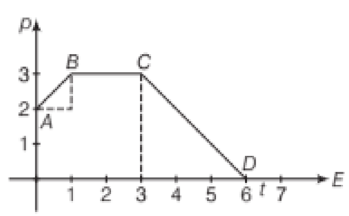
(ii) Centre of mass of a cylinder lies at its geometrical centre, i.e. at the mid-point of its axis of symmetry.

(iii) Centre of mass of a ring lies at its geometrical centre.

(iv) Centre of mass of a cube lies at its geometrical centre, i.e. at the point of intersection of its diagonals.

No, it is not necessary that centre of mass of a body lie inside it, because in some cases such as a ring, a hollow cylinder, a hollow sphere and a hollow cube centre of mass lies outside.

29. Figure shows momentum *versus* time graph for a particle moving along x-axis. In which region, force on the particle is large. Why?



Sol. Net force is given by $F_{\text{net}} = \dfrac{dp}{dt}$

Also, rate of change of momentum = slope of graph.

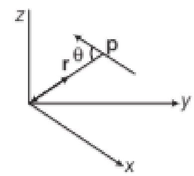
As from graph, slope AB = slope CD

And slope (BC) = slope (DE) = 0

So, force acting on the particle is equal in regions AB and CD and in regions BC and DE (which is zero).

30. Find components along x, y and z-axes of the angular momentum **L** of a particle whose position vector is **r** and momentum is **p** with components p_x, p_y and p_z. Show if the particle moves only in xy-plane, the angular momentum has only a z-component. **[NCERT]**

Sol. Angular momentum of a particle is $\mathbf{l} = \mathbf{r} \times \mathbf{p}$



Let, $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$, where (x, y, z) is the location of particle at some instant and $\mathbf{p} = p_x\hat{i} + p_y\hat{j} + p_z\hat{k}$, is the linear momentum of particle at that instant.

Then, angular momentum of particle is given by

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix}$$

$$= \hat{i}\,(p_z \cdot y - p_y \cdot z) - \hat{j}\,(p_z \cdot x - p_x \cdot z) + \hat{k}\,(p_y \cdot x - p_x \cdot y) \quad \text{...(i)}$$

So, components of angular momentum along x, y and z-axes are

$$L_x = p_z y - p_y z$$
$$L_y = p_z x - p_x z$$

and $\quad L_z = p_y x - p_x y$, respectively

Now, if particle is confined to xy-plane, then

$$\mathbf{r} = x\hat{i} + y\hat{i} \Rightarrow z = 0$$

and $\quad \mathbf{p} = p_x\hat{i} + p_y\hat{j} \Rightarrow p_z = 0$

Substituting in Eq. (i), we get

$$\mathbf{L} = \hat{k}\,(p_y x - p_x y)$$

So, angular momentum only has a z-component.

As angular momentum is cross-product of **r** and **p**.

$\therefore$ It is always perpendicular to plane containing **r** and **p**.

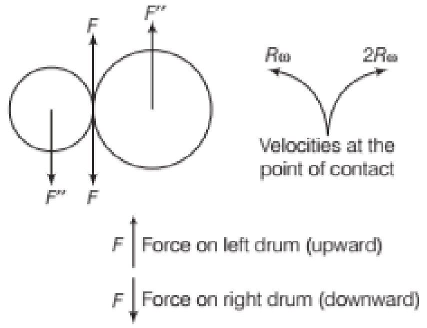
31. Two cylindrical hollow drums of radii R and $2R$, and of a common height h, are rotating with angular velocities ω (anti-clockwise) and ω (clockwise), respectively.

Their axes, fixed are parallel and in a horizontal plane separated by $(3R + \delta)$. They are now brought in contact $(\delta \to 0)$.

(i) Show the frictional forces just after contact.
(ii) Identify forces and torques external to the system just after contact.
(iii) What would be the ratio of final angular velocities when friction ceases?

[NCERT Exemplar]

Sol. (i)



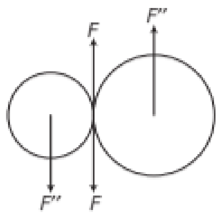
(ii) $F' = F = F''$ where F and F'' are external forces through support.

$$F_{net} = 0$$

External torque $= F \times 3R$, anti-clockwise.

(iii) Let ω_1 and ω_2 be final angular velocities (anti-clockwise and clockwise respectively).

Finally, there will be no friction.



Hence, $R\omega_1 = 2R\omega_2 \Rightarrow \dfrac{\omega_1}{\omega_2} = 2$

32. A particle on a rotating disc have initial and final angular positions are

(i) -2 rad, $+6$ rad (ii) -4 rad, -8 rad

(iii) 6 rad, -2 rad,

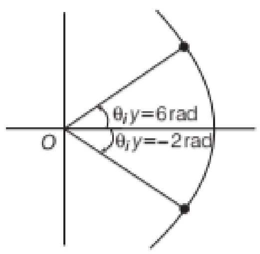
In which case, particle undergoes a negative displacement.

Sol. (i) Angular displacement is

$$\Delta\theta = \theta_f - \theta_i = 6 - (-2) = 8 \text{ rad}$$

(ii) $\Delta\theta = \theta_f - \theta_i = -8 \text{ rad} - (-4 \text{ rad}) = -4 \text{ rad}$

(iii) $\Delta\theta = -2 \text{ rad} - (+6 \text{ rad}) = -8 \text{ rad}$



In cases (ii) and (iii), angular displacement is negative.

33. A particle of mass m is projected from origin O with speed u at an angle θ with positive x-axis.

Find the angular momentum of particle at any time t about O before the particle strikes the ground again.

Sol. Let particle is at P at any instant t and its position vector is $\mathbf{r}$.

Then, $\mathbf{r} = x\,\hat{i} + y\,\hat{j}$

where, $x = u\cos\theta \cdot t$

and $y = u\sin\theta \cdot t - \dfrac{1}{2}gt^2$

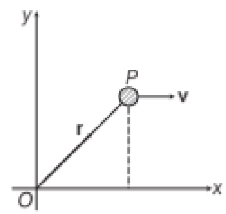
and if $\mathbf{v}$ = velocity vector of particle at that instant.

Then, $\mathbf{r} = x\hat{i} + y\hat{j}$

where, $x = u\cos\theta.t$

and $y = u\sin\theta.t - \dfrac{1}{2}gt^2$

and if $\mathbf{v}$ = velocity vector of particle at that instant.



Then, $\mathbf{v} = v_x\hat{i} + v_y\hat{j}$

where, $v_x = u\cos\theta$

and $v_y = u\sin\theta - gt$

So, $\mathbf{r} = (u\cos\theta)t\,\hat{i} + \left((u\sin\theta)\cdot t - \dfrac{1}{2}gt^2\right)\hat{j}$

and $\mathbf{v} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\,\hat{j}$

So, $\mathbf{L}$ = angular momentum of particle

$$= m(\mathbf{r} \times \mathbf{v})$$

$$= m\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u\cos\theta t & \left(u\sin\theta - \dfrac{1}{2}gt\right)t & 0 \\ u\cos\theta & (u\sin\theta - gt) & 0 \end{vmatrix}$$

$$= -\frac{1}{2}m(u\cos\theta)\,gt^2.\hat{k}$$

So, angular momentum is along z-axis.

34. A car of weight 1800 kg. The distance between its front and back axles is 1.8 m. Its centre of gravity is 1.05 m behind the front axle. Determine the force exerted by the level ground on each front wheel and each back wheel.

[NCERT Exemplar]

Moment of forces about centre of gravity is always zero.

Sol. Total mass of the car = 1800 kg

Let m and $(900 - m)$ kg be the masses of each front wheel and each back wheel, respectively.

Distance of centre of gravity from the front axle = 1.05 m

$\therefore$ Distance of centre of gravity from the back axle $= 1.80 - 1.05 = 0.75$m

Taking torque about centre of gravity,

$$m \times 1.05 = (900 - m) \times 0.75$$

or $\quad 1.05\,m + 0.75\,m = 900 \times 0.75$

or $\quad 1.80\,m = 900 \times 0.75$

or $\quad m = \dfrac{900 \times 0.75}{1.80} = 375\text{ kg}$

$\therefore \quad (900 - m) = 900 - 375 = 525\text{ kg}$

$\therefore$ Weight of each front wheel $(w_1) = m_1 g$

$$w_1 = 375 \times 9.8 = 3675\text{ N}$$

Force exerted by the level ground on each front wheel.

= force exerted by each front wheel on the level ground $(w_1) = 3675$ N

Weight of each back wheel $(w_2) = m_2 g$

$$w_2 = 525 \times 9.8 = 5145\text{ N}$$

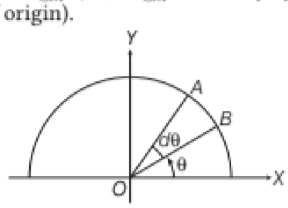
$\therefore$ Force exerted by the level ground on each back wheel.

= force exerted by each back wheel on level ground $(w_2) = 5145$ N.

LONG ANSWER Type II Questions

35. Find position of centre of mass of a semicircular disc of radius r. **[NCERT Exemplar]**

Sol. As semicircular disc is symmetrical about its one of diameter, we take axes as shown. So, now we only have to calculate Y_{CM} $\cdot$(As, X_{CM} is zero by symmetry and choice of origin).



Axis of symmetry CM lies on this

Now, $\quad Y_{CM} = \dfrac{\int y\,dm}{\int dm}$

From our text, $M \cdot Y_{CM} = \int y\,dm$

$$Y_{CM} = \frac{1}{M}\int y\,dm = \frac{\int y\,dm}{\int dm}$$

Now, for a small element OAB, as element is small and it can be treated as a triangle so,

$$\text{Area of sector } OAB = \frac{1}{2} \times r \times r d\theta$$

Height of triangle $= r$

Base of triangle $= AB = r d\theta$

So, its mass $dm = \dfrac{1}{2} r^2 d\theta \cdot \rho \qquad \left[\because \rho = \dfrac{\text{mass}}{\text{area}}\right]$

As centre of mass of a triangle is at a distance of $\dfrac{2}{3}$ from its vertex (at centroid, intersection of medians). So, $y = \dfrac{2}{3} r\sin\theta$ (location of CM of small sector AOB).

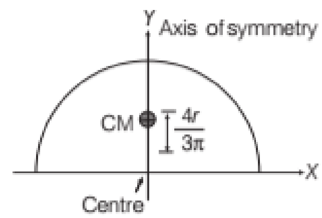
So, $Y_{CM} = \dfrac{\int_0^\pi y\,dm}{\int_0^\pi \frac{1}{2} r^2 \rho\, d\theta}$

$$= \frac{\int_0^\pi \frac{2}{3} r\sin\theta \times \frac{1}{2} r^2 d\theta \cdot \rho}{\int_0^\pi \frac{1}{2} r^2 \rho\, d\theta}$$

$$= \frac{\frac{1}{2} \times \frac{2}{3} r^3 \cdot \rho \int_0^\pi \sin\theta\, d\theta}{\frac{1}{2} r^2 \rho \cdot \int_0^\pi d\theta} = \frac{\frac{2r}{3}\int_0^\pi \sin\theta\, d\theta}{\int_0^\pi d\theta}$$

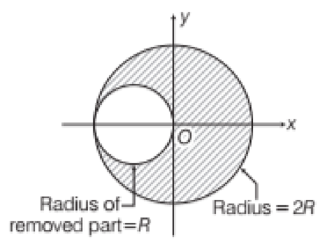
$$= \frac{\frac{2r}{3}[-\cos\theta]_0^\pi}{[\theta]_0^\pi} = \frac{\frac{-2r}{3}(\cos\pi - \cos 0^\circ)}{(\pi - \theta)}$$

$= 4r/3\pi$



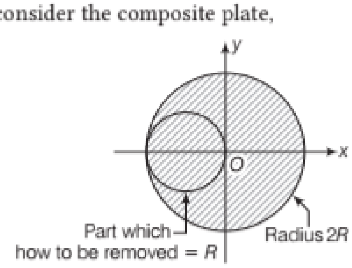
So, CM of disc is at a distance of $\dfrac{4r}{3\pi}$ from its centre on its axis of symmetry.

36. A disc of radius R is removed from a disc of radius $2R$ as shown.



Find centre of mass of above disc with hole. **[NCERT]**

Sol. First consider the composite plate,



x-coordinate of centre of mass of composite plate is given by $x_{CM} = \dfrac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$

where,

m_1 = mass of disc of radius $2R$

x_1 = position of centre of mass of disc with hole

m_2 = mass of part removed of radius R

x_2 = position of centre of mass of part removed.

Now, as centre of mass of composite plate is at origin, so

$$x_{CM} = 0 = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

or $m_1 x_1 + m_2 x_2 = 0$

Now, we are finding x_1.

So, $$x_1 = -\frac{m_2 x_2}{m_1}$$

Now, $$\frac{m_2}{m_1} = \frac{\text{density} \times \text{volume of portion 2}}{\text{density} \times \text{volume of portion 1}}$$

$$= \frac{\text{thickness} \times \text{area of portion 2}}{\text{thickness} \times \text{area of portion 1}}$$

$$\Rightarrow \frac{m_2}{m_1} = \frac{A_2}{A_1} = \frac{\pi R^2}{\pi (2R)^2 - \pi R^2} = \frac{1}{3}$$

Also, $x_2 = -R$, (negative sign is w.r.t. coordinate axes)

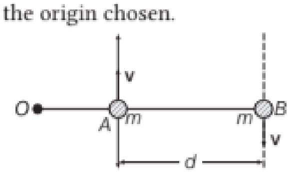
Hence, $$x_1 = -\frac{m_2}{m_1} \cdot x_2$$

$$= -\frac{1}{3}(-R) = \frac{1}{3}R$$

So, centre of mass located at $\left(x = \frac{1}{3}R,\ y = 0\right)$ point.

37. **Two particles each of mass m and speed v travel in opposite direction along parallel lines, separated by a distance d. Show that vector angular momentum of the two particles system is same whatever be the point about which angular momentum is taken.** **[NCERT]**

Sol. A sketch of system of particles at any instant is shown. Let, O be the origin chosen.



Then, angular momentum of particle at A is

$$\mathbf{l}_1 = \mathbf{OA} \times \mathbf{p} = \mathbf{OA} \times m\mathbf{v}$$
$$= m(\mathbf{OA} \times \mathbf{v})$$

and angular momentum of particle at B is

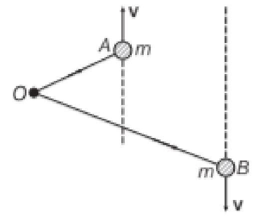
$$\mathbf{l}_2 = \mathbf{OB} \times \mathbf{p} = \mathbf{OB} \times (-m\mathbf{v})$$
$$= -m(\mathbf{OB} \times \mathbf{v})$$

So, total angular momentum of the system of particles is

$$\mathbf{L} = \mathbf{l}_1 + \mathbf{l}_2$$
$$= m(\mathbf{OA} \times \mathbf{v}) - m(\mathbf{OB} \times \mathbf{v})$$
$$= m(\mathbf{OA} \times \mathbf{v} - \mathbf{OB} \times \mathbf{v})$$
$$= m(\mathbf{OA} - \mathbf{OB}) \times \mathbf{v}$$
$$= m(\mathbf{BA}) \times \mathbf{v}$$

{As, **BA** = position vector of **A** – position vector of **B**}

Above expression is independent of choice of origin.



This is true even when particles are not in a straight line.

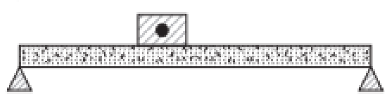
$$\mathbf{L}_i = \mathbf{l}_1 + \mathbf{l}_2 = m(\mathbf{OA} \times \mathbf{v} - \mathbf{OB} \times \mathbf{v})$$
$$= m(\mathbf{BA}) \times \mathbf{v}$$

Which is same as previous result. So, angular momentum of system is independent of choice of origin.

38. **A beam of uniform cross-section and uniform mass-density of mass 20 kg is supported at ends. A mass of 5 kg is placed at a distance of $L/5$ m from one of its end. If beam is L m long, what are reactions of supports?**

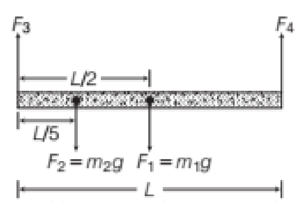
In dealing with problems of equilibrium of a rigid body (like beam in this question), first of all draw a free body diagram of the system, indicating all of the forces acting on the system.

Sol. In given case, our system looks like



Forces involved are (i) weight of beam $= F_1 = m_1 g$. (ii) weight of mass placed over beam $= F_2 = m_2 g$. (ii) Reactions of supports F_3 and F_4 which we have to found.

Free body diagram of our system is



In static equilibrium problem, calculation involves the following equations

Sum of all vertical (or vertical component of) forces is zero, $\Sigma F_V = 0$

Sum of all horizontal forces or horizontal components of forces is zero, $\Delta F_H = 0$

Sum of all torques is zero.

$$\Sigma\tau = 0$$

You may take signs as
All forces along + y-direction are positive.
All forces along + x-direction are positive.
All forces along − y-direction are negative.
All forces along −x-direction are negative.
All anti-clockwise torques are positive.
All clockwise torques are negative.
In given problem, as the beam is in equilibrium,

$$\Sigma F_V = 0$$

or $$F_3 + F_4 - F_2 - F_1 = 0$$

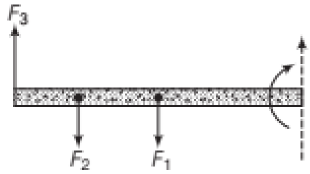
or $$F_3 + F_4 - m_2 g - m_1 g = 0 \quad \text{...(i)}$$

$$\Sigma F_H = 0$$

As there is no horizontal force involved here so, this does not give any equation.

and $$\Sigma \tau = 0$$

Let us choose rotation axis through right hand corner of beam.



Sum of all torques is

$$F_2 \times \frac{4}{5}L + F_1 \times \frac{L}{2} - F_3 \times L = 0$$

(no torque due to force F_4)
From above equation,

$$\frac{4}{5}F_2 + F_1 - F_3 = 0$$

or $$\frac{4}{5}m_2 g + m_1 g - F_3 = 0 \quad \text{...(ii)}$$

When all the equations are formed, solve them like linear simultaneous equations to get desired results.
In given problem, from Eq. (ii) we get

$$F_3 = \frac{4}{5}m_2 g + m_1 g$$

$$= \frac{4}{5} \times 5 \times 10 + 20 \times 10$$

$$= 20 \times 12 = 240 \text{ N}$$

$$m_1 = 20 \text{ kg}$$

and $$m_2 = 5 \text{ kg}$$

$$g = 10 \text{ m/s}^2$$

Substituting value of F_3, m_1 and m_2 in Eq. (i), we get

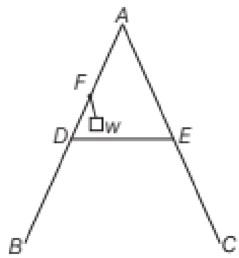
$$240 + F_4 - 5 \times 10 - 20 \times 10 = 0$$

or $$F_4 = 50 + 200 - 240$$

$$F_4 = 10 \text{ N}$$

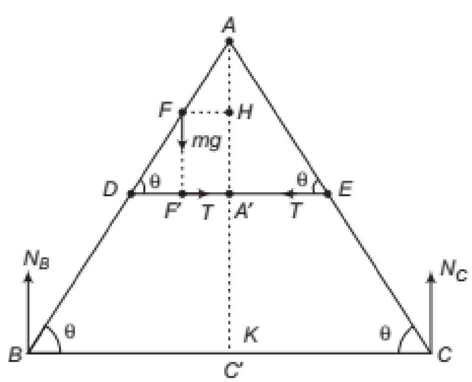
So, support reactions are 240 N on left support and 10 N on right support.

39. As shown in the figure, the two sides of a step ladder BA and CA are 1.6 m long and hinged at A. A rope DE, 0.05 m is tied half way up. A weight 40 kg is suspended from a point F, 1.2 m from B along the ladder BA. Assuming the floor to be frictionless and neglecting the weight of the ladder, find the tension in the rope and forces exerted by the floor on the ladder.
(Take, $g = 9.8 \text{ m/s}^2$) **[NCERT]**



Sol. Different forces acting on the system are

(i) Normal reaction of the floor at point B, N_B acting in vertically upward direction.
(ii) Normal reaction of the floor at point C, N_C acting in vertically upward direction.
(iii) Weight at point F, acting vertically downward.
(iv) Tensions in the string, as the ladder will have a tendency to slide at point B and point C (as the floor is frictionless) so, at both points D and E the string will be stretched in outward direction, therefore the tensions at both points will be along inward.



Given, length of each side of ladder

$$AB = AC = 1.6 \text{ m}$$

Let the sides of the ladder subtend an angle θ with the horizontal and forces exerted by the floor on the ladder at points B and C be N_B and N_C, respectively.

Length of the rope $(DE) = 0.5$ m

Weight suspended $(w) = 40$ kg-f

$$= 40 \times 9.8 = 392 \text{ N}$$

Distance $(BF) = 1.2$ m

∴ Distance $(AF) = AB - BF = 1.6 - 1.2 = 0.4$ m

$$\angle ABC = \angle ADE = \angle ACB = \angle AED = \theta$$

Let A' be the mid-point of the rope

$\therefore \quad DA' = \frac{0.5}{2} = 0.25 \text{ m}$

$\therefore \quad DF' = F'A' = \frac{1}{2} DA' = \frac{1}{2} \times 0.25 = 0.125 \text{ m}$

For translational equilibrium of the step ladder,

$$N_B + N_C = W$$

$$N_B + N_C = 392 \text{ N} \quad \text{...(i)}$$

Taking moment of forces about point A for side AB,

$$N_B \times BC' = W \times F'A' + T \times AA'$$

But $\quad BC' = AB\cos\theta$

and $\quad AA' = AD\sin\theta$

$$N_B \times AB\cos\theta = W \times 0.125 + T \times AD\sin\theta \quad \text{...(ii)}$$

But from $\Delta DFF'$,

$$\cos\theta = \frac{DF'}{DF} = \frac{0.125}{0.4} = 0.3125$$

$$\theta = 72.8°$$

$\therefore \quad \cos\theta = \cos 72.8° = 0.3125$

$$\sin\theta = \sin 72.8° = 0.9553$$

$$\tan\theta = \tan 72.8° = 3.2350$$

Substituting values in Eq. (ii), we get

$$N_B \times 1.6 \times 0.3125 = (392 \times 0.125) + (T \times 0.8 \times 0.9553)$$

or $\quad 0.5\, N_B = 0.764T + 49 \quad \text{...(iii)}$

Now, taking moment of forces about point A, for side AC.

$$N_C \times CC' = T \times AA'$$

But $CC' = AC\cos\theta$ and $AA' = AE\sin\theta$

$\therefore \quad N_C \times AC\cos\theta = T \times AE\sin\theta$

$$N_C \times 1.6 \times 0.3125 = T \times 0.8 \times 0.9553$$

$$0.5\, N_C = 0.764\, T \quad \text{...(iv)}$$

Substituting value from Eq. (iv) in Eq. (iii), we get

$$0.5\, N_B = 0.5\, N_C + 49$$

$$0.5\,(N_B - N_C) = 49$$

$$\frac{1}{2}(N_B - N_C) = 49$$

$$N_B - N_C = 98 \quad \text{...(v)}$$

Adding Eqs. (i) and (v), we get

$$2N_B = 392 + 98 = 490$$

or $\quad N_B = 245 \text{ N}$

From Eq. (i),

$$N_C = 392 - 245 \text{ N} = 147 \text{ N}$$

From Eq (iv), $\quad T = \frac{0.5 \times 147}{0.764} = 96.2 \text{ N}$

40. From a uniform disc of radius R, a circular section of radius $\frac{R}{2}$ is cut out. The centre of the hole is at $\frac{R}{2}$ from the centre of the original disc. Locate the centre of gravity of the resulting flat body. **[NCERT]**

Sol. Let mass per unit area of the disc be m.

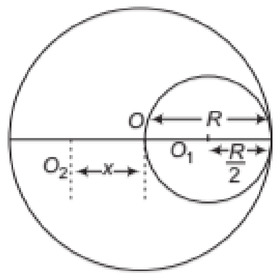
$\therefore$ Mass of the disc (M) = total area of disc × mass per unit area $= \pi R^2 m$.

Mass of the portion removed from the disc (M')

$$= \pi\left(\frac{R}{2}\right)^2 m$$

$$\Rightarrow \quad \frac{\pi R^2}{4} m = \frac{M}{4}$$

The centre of mass of the original disc is O and the centre of mass of the removed part is O_1 and let centre of mass of the remaining part be O_2.



According to the question, figure can be drawn as

Here, $\quad OO_1 = \frac{R}{2}$

The remaining portion of the disc can be considered as a system of two masses M at O and $-M' = -\frac{M}{4}$ at O_1.

If the distance of the centre of mass of the remaining part from the centre O is at a distance x, then

$$x = \frac{M \times 0 - M' \times \frac{R}{2}}{M - M'} = \frac{-\frac{M}{4} \times \frac{R}{2}}{M - \frac{M}{4}}$$

$$= -\frac{MR}{8} \times \frac{4}{3M} = -\frac{R}{6}$$

Therefore, centre of mass of the remaining part is at $\frac{R}{6}$ to the left of centre O.

ASSESS YOUR TOPICAL UNDERSTANDING

OBJECTIVE Type Questions

1. A system of particles is called a rigid body when
   (a) any two of particles of system may have displacements in opposite directions under action of a force.
   (b) Any two of particles of system may have velocities in opposite directions under action of a force.
   (c) Any two particles of system may have a non-zero relative velocity.
   (d) Any two of particles of system may have displacements in same direction under action of a force.

2. For n-particles in a space, the suitable expression for the position vector of centre of mass is
   (a) $\frac{\Sigma m_i \mathbf{r}_i}{m_i}$ (b) $m_i \mathbf{r}_i$
   (c) $\frac{\Sigma m_i \mathbf{r}_i}{M}$ (d) $\frac{m_i \mathbf{r}_i}{m_i}$

3 For which of the following does the centre of mass lie outside the body? **[NCERT Exemplar]**
   (a) A pencil
   (b) A shotput
   (c) A dice
   (d) A bangle

4. Angular velocity vector is directed along
   (a) the tangent to the circular path
   (b) the inward radius
   (c) the outward radius
   (d) the axis of rotation

5. In translational equilibrium,
   (a) $\sum_{i=1}^{n} \mathbf{F}_{net} = 0$
   (b) $\sum_{i=1}^{n} \tau_{net} = 0$
   (c) Both (a) and (b) are the necessary conditions for the translational equilibrium
   (d) Particle may be in equilibrium when (a) and (b) are not fulfilled.

Answer

1. (c)	2. (c)	3. (d)	4. (d)	5. (a)

VERY SHORT ANSWER Type Questions

6. What is the basic difference between pure translation motion and pure rotational motion?
7. A ball is moving with a velocity of 10 m/s. Suddenly, it breaks into two equal parts. What will be the speed of centre of mass of the system after explosion? **[Ans.** 10 m/s]
8. A particle is moving on the x-axis with a linear velocity of 2 m/s. What will be its angular momentum about the origin? **[Ans.** zero]

SHORT ANSWER Type Questions

9. It is difficult to open the door by pushing it or pulling it at the hinge. Why?
10. Which physical quantities are expressed by the following:
    (i) the rate of change of angular momentum?
    (ii) moment of linear momentum?
11. If no external torque acts on a body, will its angular velocity remains conserved?
12. Why is it easier to open a tap with two finger than with one finger?
13. Can the couple acting on a rigid body produce translatory motion?
14. Can be a body in translational motion have angular momentum? Why?
15. If a body is rotating, is it necessarily being acted upon by an external torque?

LONG ANSWER Type I Questions
|3 Marks|

16. Locate the centre of mass of uniform triangular lamina and a uniform cone.
17. Three identical spheres each of radius r and mass m are placed touching each other on a horizontal floor. Locate the position of centre of mass of the system. **[Ans.** $\left(r, \frac{r}{3}\right)$]
18. Explain the concept of torque. Obtain an expression for torque in polar coordinates.

LONG ANSWER Type II Questions

19. In given pulley mass system, mass $m_1 = 500$ g, $m_2 = 460$ g and the pulley has a radius of 5 cm. When released from rest, heavier mass falls through 7.50 cm in 5 s. There is no slippage between pulley and string.

(i) What is magnitude of acceleration of masses?

(ii) What is magnitude of pulley's angular acceleration?

[**Ans.** $a = 6 \times 10^{-2}$ m/s^2, $\alpha = 1.20$ rad/s^2]

20. Find the centre of mass for a solid cone of base radius r and height h.

[**Ans.** At height $\frac{h}{4}$ from the point O. O is at the centre of the base of the cone]

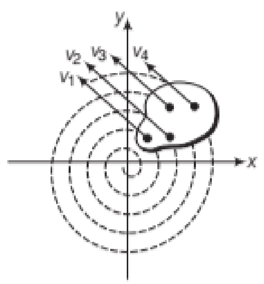
21. A metal 13 bar 70 cm long and 4 kg in mass supported on two knife edges placed 10 cm from each end. A 6 kg weight is suspended at 30 cm from one end. Find the reactions at the knife edges. Assume the bar to be of uniform cross-section and homogeneous. [**Ans.** 55 N and 43 N]

|TOPIC 2|
Moment of Inertia and Rolling Motion

MOMENT OF INERTIA

The property of a body by virtue of which it opposes the torque tending to change its state of rest or of uniform rotation about an axis is called **rotational inertia** or **moment of inertia.**

To understand moment of inertia, let us consider a rigid body rotating about a fixed axis. For a rotating rigid body, kinetic energy is the sum of kinetic energies of individual particles.



Rotation of a rigid body

So, kinetic energy of a rotating body,

$$K = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + \cdots + \frac{1}{2}m_nv_n^2$$

or $$K = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

Substituting $v = r\omega$, $K = \Sigma \frac{1}{2} m_i r_i^2 \omega^2$

where m_i is the mass of particle, v_i is the linear velocity.

As, the angular velocity ω is same for all the particles of rigid body.

So, $$K = \frac{1}{2}\omega^2 \left(\sum_{i=1}^{n} m_i r_i^2\right) \qquad \ldots\text{(i)}$$

The quantity $\Sigma (m_i r_i^2)$ depends on distribution of mass around axis of rotation. This quantity is called **moment of inertia** (I).

$$\boxed{\text{Kinetic energy, } K = \frac{1}{2}I\omega^2} \qquad \ldots\text{(ii)}$$

For translational motion, the kinetic energy of a particle of mass m and speed v is

$$\text{KE} = \frac{1}{2}mv^2 \qquad \ldots\text{(iii)}$$

By comparing Eqs. (ii) and (iii), we can get that moment of inertia in the rotational analogue of mass.

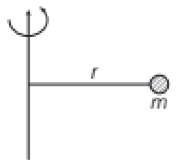
So, for a rotating body, its moment of inertia

$$I = \sum_{i=1}^{n} m_i r_i^2$$

Hence, moment of inertia of a rigid body about a fixed axis is defined as the sum of the products of the masses of the particles constituting the body and the squares of their respective distances from the axis of rotation.

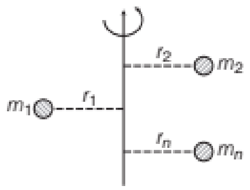
Some Important Points about Moment of Inertia

- Moment of inertia is a measure of rotational inertia and it plays same role as played by mass in translational motion.
- So, for a particle of mass m revolving in a path of radius r, moment of inertia is $I = mr^2$.



Particle revolving in a path

- For a system of n particles, $I = \sum_{i=1}^{n} m_i r_i^2$

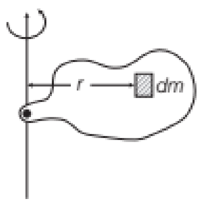


Number of particles revolving in a path

- For continuous mass distributions, treating a differentially small mass as a particle moment of inertia of complete body

$$I = \int r^2 dm$$

where, integral is taken for complete body.



A small particle in a rigid body to find moment of inertia of the whole body

- A heavy wheel called flywheel is attached to the shaft of steam engine, automobile engine etc., because of its large moment of inertia, the flywheel opposes the sudden increase or decrease of the speed of the vehicle. It allows a gradual change in the speed and prevents jerky motion and hence ensure smooth ride of passengers.
- In a bicycle, bullock-cart etc., the moment of inertia is increased by concentrating most of the mass at the rim of the wheel and connecting the rim to the axle through the spokes.
Even after we stop paddling, the wheels of a bicycle continue to rotate for sometime due to their large moment of inertia.

Relation between Rotational Kinetic Energy and Moment of Inertia

We have rotational kinetic energy

$$\text{KE} = \frac{1}{2}(\Sigma mr^2)\,\omega^2$$

$$\because \quad \text{KE} = \frac{1}{2}mv^2 = \frac{1}{2}mr^2\omega^2$$

For system of n particles, $\text{KE} = \frac{1}{2}(\Sigma\, mr^2)\,\omega^2$

But $\quad \Sigma\, mr^2 = I$

Therefore, $\quad$ rotational $\text{KE} = \frac{1}{2} I\omega^2$

When $\omega = 1$, rotational $\text{KE} = \frac{1}{2} I$

or $\quad$ **Moment of inertia, $I = 2 \times$ rotational KE**

Hence, **the moment of inertia of a rigid body about an axis of rotation is numerically equal to twice the rotational kinetic energy of the body when it is rotating with unit angular velocity about that axis.**

EXAMPLE |1| Increasing the Revolution

An energy of 484 J is spent in increasing the speed of a flywheel from 60 rpm to 360 rpm. Calculate moment of inertia of flywheel.

Sol. Energy spent, $W = 484$ J

Initial speed, $\omega_1 = 60 \text{ rpm} = \frac{60}{60} \times 2\pi = 2\pi$ rad/s

Final speed, $\omega_2 = 360 \text{ rpm} = \frac{360}{60} \times 2\pi = 12\pi$ rad/s

Moment of inertia, $I = ?$

Energy spent, $W = E_2 - E_1 = \frac{1}{2} I\omega_2^2 - \frac{1}{2} I\omega_1^2$

$$= \frac{1}{2} I[(12\pi)^2 - (2\pi)^2] = 70 I\pi^2$$

$$\Rightarrow \quad I = \frac{484}{70 \times \pi^2} = 0.7 \text{ kg-m}^2.$$

Radius of Gyration (K)

The radius of gyration of a body about an axis may be defined as the distance from the axis of a mass point whose mass is equal to the mass of whole body and whose moment of inertia is equal to the moment of inertia of the body about the axis. When square of radius of gyration is multiplied with the mass of the body, gives the moment of inertia of the body about the given axis.

$$I = MK^2$$

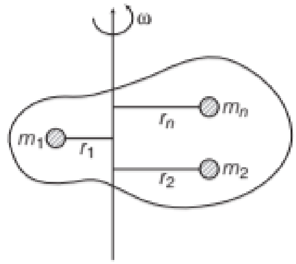
$$\boxed{\text{Radius of gyration, } K = \sqrt{\frac{I}{M}}}$$

where, K is radius of gyration of the body.

If we consider a rotating body, then its moment of inertia is

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + \cdots + m_n r_n^2$$

Let $\quad m_1 = m_2 = m_3 = \cdots = m_n = m,$



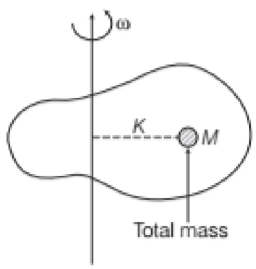
A rotating body with various masses

Then, $\quad I = mr_1^2 + mr_2^2 + mr_3^2 + \cdots + mr_n^2$

$$= m(r_1^2 + r_2^2 + r_3^3 + \cdots + r_n^2) \quad \ldots \text{(i)}$$

and by definition of radius of gyration,

$$I = mK^2 \quad \ldots \text{(ii)}$$



A rotating body

So, from Eqs. (i) and (ii), we get

$$MK^2 = m(r_1^2 + r_2^2 + \cdots + r_n^2)$$

or $$K^2 = \frac{m}{M}(r_1^2 + r_2^2 + \cdots + r_n^2)$$

$$K^2 = \frac{m}{mn}(r_1^2 + r_2^2 + \cdots r_n^2) \quad [\text{as } M = mn]$$

or $$\boxed{K = \sqrt{\frac{r_1^2 + r_2^2 + \cdots + r_n^2}{n}}}$$

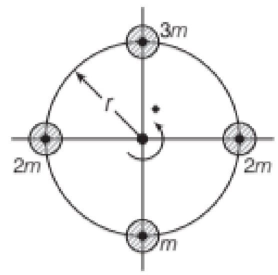
Hence, **radius of gyration of a body about a given axis is equal to root mean square distance of constituent particles from the given axis.**

Note

- Factors on which radius of gyration of a body depend
- Position and direction of the axis of rotation.
- Distribution of mass about the axis of rotation.

EXAMPLE |2| Radius of Gyration when a number of Masses having Rotation

Four masses are rotated about an axis as shown. Find radius of gyration of the system. **[NCERT]**



Sol. Moment of inertia of a mass about axis

$$= \text{mass} \times (\text{distance})^2$$

$\therefore$ Moments of inertia of masses are

$$I_1 = 2mr^2$$

$\Rightarrow$ $$I_2 = 3m \cdot r^2$$

$$I_3 = 2m \cdot r^2$$

$\Rightarrow$ $$I_4 = m \cdot r^2$$

So, total moment of inertia of system of masses

$$= I_1 + I_2 + I_3 + I_4 = 8\,mr^2$$

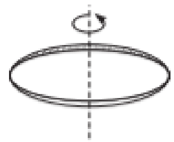
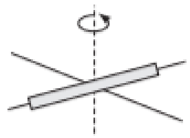
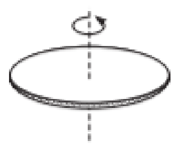
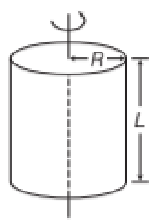
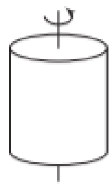
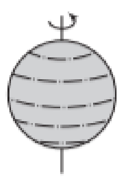
Total mass of system = $8m$

If radius of gyration is K, then $I = MK^2$

or $$K^2 = \frac{I}{M} = \frac{8\,mr^2}{8\,m}$$

or $$K = r$$

Moment of Inertia in Some Standard Cases

Body	Figure	Moment of inertia	K	K^2/R^2
Thin circular ring, radius R		MR^2	R	1
Thin rod length, L		$\frac{1}{12}ML^2$	$\frac{L}{\sqrt{12}}$	
Circular disc, radius R		$\frac{1}{2}MR^2$	$\frac{R}{2}$	$\frac{1}{2}$
Hollow cylinder, radius R		MR^2	R	1
Solid cylindrical, radius R		$\frac{MR^2}{2}$	$\frac{R}{2}$	$\frac{1}{2}$
Solid sphere, radius R		$\frac{2}{5}MR^2$	$\sqrt{\frac{2}{5}}R$	$\frac{2}{5}$

KINETIC EQUATIONS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

The three equations of linear motion are

(i) $v = u + at$

(ii) $s = ut + \frac{1}{2}at^2$

(iii) $v^2 - u^2 = 2as$

where, the symbols have their usual meaning.

In the similar way, we can write equations of motion for rotational motion such as

(i) $\omega = \omega_0 + \alpha t$ (ii) $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

(iii) $\omega^2 - \omega_0^2 = 2\alpha(\theta - \theta_0)$

where, θ_0 and ω_0 are the initial angular displacement and initial angular velocity of the body, respectively.

EXAMPLE |3| Angular Speed of Wheels

The angular speed of a motor wheel is increased from 1200 rpm to 3120 rpm in 16s.

(i) What is its angular acceleration, assuming the acceleration to be uniform?

(ii) How many revolutions does the engine make during this time? **[NCERT]**

Sol. (i) Given,

Initial revolution of the wheel, $N_0 = 1200$ rpm

Final revolution of the wheel, $N = 3120$ rpm

Time, $t = 16$ s

Angular acceleration, $\alpha = ?$

Number of revolutions of the engine = ?

Initial angular speed, $\omega_0 = \frac{2\pi N_0}{60} = \frac{2\pi \times 1200}{60}$

$= 40\pi$ rad/s

Final angular speed, $\omega = \frac{2\pi N}{60} = \frac{2\pi \times 3120}{60} = 104\pi$ rad/s

Angular acceleration, $\alpha = \frac{\omega - \omega_0}{t}$,

$$\alpha = \frac{104\pi - 40\pi}{16} = 4\pi \text{ rad/s}^2$$

(ii) Now, evaluate the angular displacement of the motor wheel because the number of revolutions of the wheel of the engine is

$$N_E = \frac{\theta}{2\pi}$$

The angular displacement in time t is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\theta = 40\pi \times 16 + \frac{1}{2} \times 4\pi \times (16)^2 = 1152\pi \text{ rad}$$

Determine the number of revolutions done by the wheel of the engine in time, $t = 16$ s.

$$N_E = \frac{\theta}{2\pi} = \frac{1152\pi}{2\pi}$$

$$N_E = 576$$

The revolutions per second can also be find out as

$$\frac{576}{16} = 36 \text{ rps}$$

EXAMPLE |4| A Constant Angular Acceleration of a Wheel

A wheel has a constant angular acceleration of 4.2 rad/s^2. During a certain 8.05 s interval, it turns through angle of 140 rad. Assuming that wheel started from rest, how long it had been in motion before the start of the 8.0 s?

Sol. Let ω_0 = initial angular speed at $t = 0$

Then, angle turned at end of 8.0 s is 140°.

Using $\theta = \omega_0 t + \frac{1}{2}\alpha t^2$

We have $\omega_0 = \frac{\theta - \frac{1}{2}\alpha t^2}{t}$

$$\omega_0 = \frac{140 - \frac{1}{2}(4.2)(8.0)^2}{8.0}$$

$$\omega_0 = 0.7 \text{ rad/s}$$

Now, using $\omega = \omega_0 + \alpha t$ and taking $\omega = 0$.

We have, $t = \frac{\omega - \omega_0}{\alpha} = \frac{0 - \omega_0}{4.2}$

$$= -\frac{0.7}{4.2} = -0.16 \text{ s}$$

So, wheel starts from rest 0.16 s before.

DYNAMICS OF ROTATIONAL MOTION ABOUT A FIXED AXIS

From the given table, we compare linear motion and rotational motion about a fixed axis, i.e. z-axis.

Comparison of Translational and Rotational Motion

Pure translation	Pure rotational
Position, x	Angular position, θ
Velocity, $v = \frac{dx}{dt}$	Angular velocity, $\omega = \frac{d\theta}{dt}$
Acceleration, $a = \frac{dv}{dt}$	Angular acceleration, $\alpha = \frac{d\omega}{dt}$
Mass, m	Rotational inertia, I

Pure translation	Pure rotational
Newton's second law, $F = ma$	Newton's second law, $\tau = I\alpha$
Work done, $W = \int F\,dx$	Work done, $W = \int \tau\,d\theta$
Kinetic energy, $K = \frac{1}{2}mv^2$	Kinetic energy, $K = \frac{1}{2}I\omega^2$
Power, $P = Fv$	Power, $P = \tau\omega$
Linear momentum, $p = mv$	Angular momentum, $L = I\omega$

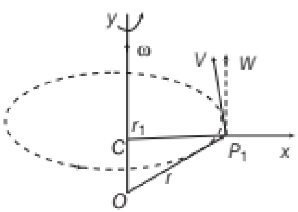
In case of rotational motion about a fixed axis, we have

(i) We need to consider only those forces that lie in planes perpendicular to the axis. The forces which are parallel to the axis of rotation will give torques perpendicular to axis. As the axis is fixed, we will ignore torques perpendicular to the axis.

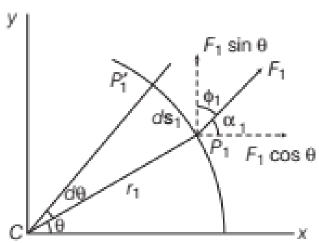
(ii) We consider only those components of the position vectors which are perpendicular to the axis of rotation. Components of position vectors along the axis will result in torques perpendicular to axis and thus be ignored.

Work Done by a Torque

Consider a rigid body rotating about a fixed axis. Let F_1 be the force acting on a particle of the body at point P_1 with its line of action in a plane perpendicular to the axis.



(a) The angular velocity vector ω is directed along the fixed axis



(b) Work done by a force acting on a particle of a body

The particle at P_1 describes a circular path of radius r_1 with centre C on the axis, $CP_1 = r_1$. At time Δt_1 the point moves to the position P_1' and displacement of the particle ds_1.

Work done by the force on the particle is given by

$$dW_1 = \mathbf{F}_1 \cdot \mathbf{ds}_1 = F_1 ds_1 \cos\phi_1 = F_1 (r_1 d\theta) \sin\alpha_1$$

where, ϕ_1 is the angle between $\mathbf{F}_1$ and the tangent at P_1 and α_1 is the angle between $\mathbf{F}_1$ and radius vector $\mathbf{OP}_1$; $\phi_1 + \alpha_1 = 90°$.

Torque due to $\mathbf{F}_1$ about the origin is given by

$$\tau_1 = \mathbf{OP}_1 \times \mathbf{F}_1$$

From Fig. (a) $OP_1 = OC + CP_1$ but OC is along the axis, therefore the torque resulting from it is excluded from our consideration.

Now, the effective torque due to F_1 is given by

$$\tau_1 = \mathbf{CP} \times \mathbf{F}_1$$

$$\therefore \quad dW_1 = \tau_1 d\theta$$

If there are more than one forces acting on the body, then work done is given by

$$dW = (\tau_1 + \tau_2 + \tau_3 + \ldots) d\theta$$

$$= \tau\, d\theta \qquad [\because \tau_1 + \tau_2 + \tau_3 + \ldots = \tau]$$

Dividing both sides by dt, we get

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt}$$

$$\Rightarrow \quad \boxed{\text{Instantaneous power, } P = \tau W}$$

The above expression indicates **instantaneous power.**

Relation between Torque and Moment of Inertia

The rotational kinetic energy of a rigid body is represented by

$$\text{KE} = \frac{1}{2} I\omega^2$$

If the body has an angular acceleration α, its rotational kinetic energy will be change.

We know $P = \frac{d}{dt}(\text{KE})$

[rate of change of KE is momentum]

$$= \frac{d}{dt}\left(\frac{1}{2} I\omega^2\right)$$

$$= \frac{1}{2} I \frac{d}{dt}(\omega^2) \qquad [\text{if } I \text{ is constant}]$$

$$= \frac{1}{2} I\, (2\omega) \frac{d\omega}{dt}$$

$$P = I\omega\alpha \qquad \left[\because \alpha = \frac{d\omega}{dt}\right]$$

$$\tau\omega = I\,\omega\alpha \qquad [\because P = \tau\omega]$$

$$\boxed{\text{Torque, } \tau = I\alpha} \qquad [I \text{ is constant}]$$

It is similar to the expression of Newton's second law for translational motion with constant mass.

EXAMPLE |5| Cord Wounded Round a Flywheel

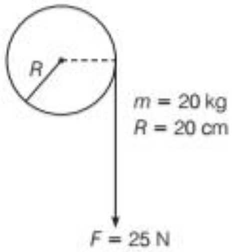
A cord of negligible mass is wound round the rim of a flywheel of mass 20 kg and radius 20 cm. A steady pull of 25 N is applied on the cord as shown in the figure. The flywheel is mounted on a horizontal axle with frictionless bearings.

(i) Compute angular acceleration of the flywheel.

(ii) Find the work done by the pull, when 2 m of the cord is unwound.

(iii) Find also the KE of the flywheel at this point. Assume that the flywheel starts from rest.

(iv) Compare the answers of parts (ii) and (iii).



Sol. (i) Torque, $\tau = FR = 25 \times 0.20 = 5$ N-m

[$\because R = 20$ cm $= 0.2$ m]

Moment of inertia, $I = \frac{MR^2}{2} = \frac{20 \times (0.2)^2}{2} = 0.4$ kg-m^2

Angular acceleration, $\alpha = \frac{\tau}{I} = \frac{5}{0.4} = 12.5$ rad/s^2

(ii) Work done by the pull, $W = F \times s = 25 \times 2 = 50$ J

(iii) $KE = \frac{1}{2} I\omega^2$

$\omega^2 = \omega_0^2 + 2\alpha\theta = 0 + 2 \times 12.5 \times 10$

$[\because \omega_0 = 0, \theta = \frac{2}{0.2} = 10 \text{ rad}]$

$\therefore \quad KE = \frac{1}{2} \times 0.4 \times 250 = 50$ J

(iv) From parts (*ii*) and (*iii*), KE = W

$\therefore$ No loss of energy due to friction.

EXAMPLE |6| Work Done by a Rotor

To maintain a rotor at a uniform angular speed of 200 rads^{-1}, an engine needs to transmit a torque of 180 N-m. What is the power required by engine? Assume 100% efficiency of engine.

Sol. Work done by torque in turning rotor by angle $d\theta$ is

$= \tau d\theta$

So, power delivered by engine

$$P = \frac{\text{Work done}}{\text{Time taken}} = \tau \frac{d\theta}{dt}$$

[dt = time for turning by angle $d\theta$.]

or $\quad P = \tau\omega$

So, power required $= 180 \times 200 = 36000$ W

$= 36$ kW $\quad$ [1 kW = 1000 W]

ANGULAR MOMENTUM IN CASE OF ROTATION ABOUT A FIXED AXIS

The general expression for total angular momentum of a system of particles is given by

$$\mathbf{L} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{p}_i = \sum_{i=1}^{n} \mathbf{r}_i \times m_i \times \mathbf{v}_i$$

$$= \sum_{i=1}^{n} \mathbf{r}_i \times m_i \times (\omega \times \mathbf{r}_i)$$

Using vector triple product, we get

$$\mathbf{L} = \sum_{i=1}^{n} m_i [(\mathbf{r}_i \cdot \mathbf{r}_i)\omega - (\mathbf{r}_i \cdot \omega) \cdot \mathbf{r}_i]$$

$$= \sum_{i=1}^{n} m_i [r_1^2 \omega - 0] = \sum_{i=1}^{n} m_i r_i^2 \omega$$

For any individual particle, the vectors of angular momentum and angular velocity are not necessarily parallel. In case of linear motion, linear momentum and linear velocity vectors are always parallel to each other.

In z-axis, the moment of inertia $I = \sum_{i=1}^{n} m_i \cdot r_1^2$ and substituting in the above equation, we get

$$\mathbf{L} = L_z = I\omega \hat{\mathbf{k}}$$

Differentiating w.r.t. time, we get

$$\frac{d}{dt}\mathbf{L}_z = \frac{d}{dt}(I\omega)\hat{\mathbf{k}} \quad \text{...(i)}$$

For rotation about fixed axis, we have $\frac{d\mathbf{L}_z}{dt} = \tau \hat{\mathbf{k}}$ ...(ii)

From Eqs. (i) and (ii), we have

$$\frac{d}{dt}(I\omega) = \tau \quad \text{...(iii)}$$

If moment of inertia does not change with time, the Eq. (ii) can be written as

$$\frac{Id\omega}{dt} = \tau \text{ or } I\alpha = \tau$$

Torque, $\tau = I\alpha$

EXAMPLE |7| A Rotating Table

A child stands at centre of a turntable with his arms out stretched. The turntable is set rotating with an angular speed of 40 rev/min.

(i) How much is the speed of child if he folds his hands and thereby reduces his MI to 2/5 times of initial value? Assume turntable is frictionless.

(ii) Show that child's new kinetic energy of rotation is more than the initial kinetic energy. How do you account for this increase in kinetic energy? **[NCERT Exemplar]**

Sol. (i) As no external torque is involved with child + turntable system, so angular momentum of system remains constant.

or $\quad I_i\omega_i = I_f\omega_f$

$I_i\omega_i$ = initial angular momentum, $I_f\omega_f$ = final angular momentum

Here, $\quad \omega_i = 40$ rpm $\Rightarrow I_f = \frac{2}{5} I_i$

Substituting, we get

$$I_i \times 40 = \frac{2}{5} I_i \times \omega_f \text{ or } \omega_f = \frac{5 \times 40}{2} = 100 \text{ rpm}$$

(ii) Initial KE $= \frac{1}{2} I_i \omega_i^2 = \frac{1}{2} \times I_i \times (40)^2$

$$= \frac{1}{2} \times I_i \times 1600 = 800\, I_i$$

Final KE $= \frac{1}{2} I_f \omega_f^2 = \frac{1}{2} \times \frac{2}{5} I_i \times (100)^2$

$$= \frac{1}{2} \times \frac{2}{5} \times I_i \times 100 \times 100 = 2000\, I_i$$

Clearly, final KE > initial KE. This energy is obtained from conversion of muscular work in KE. Muscular work has to be done in folding of arms.

EXAMPLE |8| Conservation of Angular Momentum

A comet revolves around the Sun in a highly elliptical orbit having a minimum distance of 7×10^{10} m and a maximum distance of 1.4×10^{13} m. If its speed while nearest to the sun is 60 kms^{-1}, find its linear speed when situated farthest from the sun.

Sol. Let mass of comet be M and its angular speed be ω when situated at a distance r from the Sun, then its angular momentum $L = I\omega = Mr^2\omega$

If v be the linear speed, then $L = Mr^2\omega = Mrv$

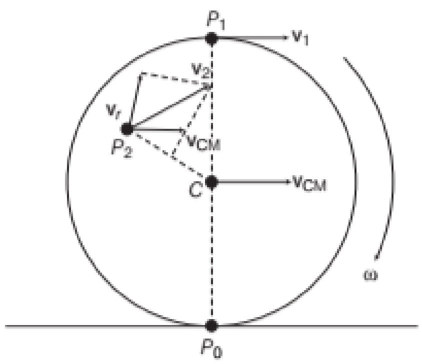
In accordance with conservation law of angular momentum, we can write that $mr_1v_1 = mr_2v_2$

$$\therefore \quad v_2 = \frac{r_1 v_1}{r_2} = \frac{7 \times 10^{10} \times 60}{1.4 \times 10^{13}} = 0.3 \text{ km/s or } 300 \text{ m/s}$$

ROLLING MOTION

The rolling motion can be regarded as the combination of pure rotation and pure translation. It is also one of the most common motions observed in daily life.

e.g. The wheels of all vehicles running on a road having rolling motion. The translational motion of a system of particles is the motion of its centre of mass.



Rolling motion of a disc on a level surface

Suppose the rolling motion (without slipping) of a circular disc on a level surface. At any instant, the point of contact P_0 of the disc with the surface is at rest (as there is no slipping). If $\mathbf{v}_{CM}$ is the velocity of centre of mass which is the geometric centre C of the disc, then the translational velocity of disc in $\mathbf{v}_{CM}$ is parallel to the level surface.

If the rotation of disc is about its symmetric axis, it passes through C. Then, velocity $\mathbf{v}_2$ at point P_2 of disc is vector sum of $\mathbf{v}_{CM}$ and $\mathbf{v}_r$ which is linear velocity on account of rotation.

The magnitude of linear velocity is

$$\mathbf{v}_r = r\omega$$

The velocity $\mathbf{v}_r$ is directed perpendicular to r. The velocity $\mathbf{v}_2$ of point P_2 as the resultant $\mathbf{v}_{CM}$ and $\mathbf{v}_r$. At P_0, the linear velocity $\mathbf{v}_r$ due to rotation is directed opposite to the translational velocity $\mathbf{v}_{CM}$. The magnitude of $\mathbf{v}_r$ is P_0 is $R\omega$, where R is the radius of the disc. As P_0 is instantaneously at rest. So,

$$\boxed{\text{Velocity of centre of mass, } \mathbf{v}_{CM} = R\omega}$$

Thus, for the disc to roll without slipping, the essential condition is $\mathbf{v}_{CM} = R\omega$.

At top of the disc, $v_1 = \mathbf{v}_{CM} + R\omega = 2\mathbf{v}_{CM}$ and is directed parallel to the surface of level.

Kinetic Energy of a Rolling Body

The kinetic energy of a body rolling without slipping is the sum of kinetic energies of translational and rotational motion.

$\therefore$ Total KE of a rolling body

$$= \text{Rotational KE} + \text{Translational KE}$$

$$= \frac{1}{2} I\omega^2 + \frac{1}{2} mv_{CM}^2$$

where v_{CM} is the velocity of centre of mass and I is the moment of inertia about the symmetry axis of the rolling body. If R is the radius and K is the radius of gyration of the rolling body, then,

$$v_{CM} = R\omega$$

and $$I = mK^2$$

$$\therefore \quad K = \frac{1}{2} mv_{CM}^2 + \frac{1}{2} mK^2 \left[\frac{v_{CM}}{R}\right]^2$$

$$\boxed{\text{Kinetic energy, } K = \frac{1}{2} mv_{CM}^2 \left[1 + \frac{K^2}{R^2}\right]}$$

EXAMPLE |9| A Solid Sphere on an Inclined Plane

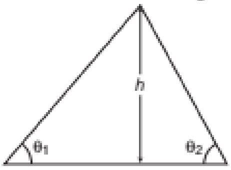
A solid sphere rolls down two different inclined planes of same height but different angles of inclination

(i) Will it reach the bottom with the same speed in each case?

(ii) Will it take longer to roll down one plane than the other?

(iii) If so, then which one? [NCERT Exemplar]

Sol. (i) Yes, (ii) Yes, (iii) On smaller inclination



(i) Let mass of sphere = m

Radius of sphere = r

Height of inclined plane = h

At top, total energy is potential energy = mgh

At bottom, the sphere has both rotational and translational kinetic energies.

Hence, total energy at bottom of incline

$$= \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

Here, $I = \frac{2}{5}mr^2$ and $v = r\omega$

So, total energy at bottom

$$= \frac{1}{2}m(r\omega)^2 + \frac{1}{2} \times \frac{2}{5}mr^2 \times \omega^2$$

$$= \frac{7}{10}mr^2\omega^2 = \frac{7}{10}mv^2$$

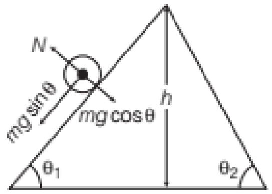
As sphere is not slipping, no energy is lost in rolling down the plane

So, equating energy at bottom and at top

$$\frac{7}{10}mv^2 = mgh \quad \text{or} \quad v = \sqrt{\frac{10}{7}gh}$$

Hence, velocity of sphere depends only on height h and acceleration due to gravity g. So, velocity at bottom is same in both cases.

(ii) When sphere rolls down the plane, forces acting on sphere are as shown



As only force down the plane is $mg \sin \theta$, acceleration down the plane is $g \sin \theta$.

If $\theta_2 > \theta_1$

Then, $g \sin \theta_2 > g \sin \theta_1$ or $a_2 > a_1$

Now, using $v = u + at$

and taking $u = 0$, $v = at$ or $t = \frac{v}{a}$

(iii) As v is same in both cases, $t \propto \frac{1}{a}$. As $a_2 > a_1$,

so $t_2 > t_1$.

Hence, sphere will take a longer time to reach the bottom of inclined plane having smaller inclination.

TOPIC PRACTICE 2

OBJECTIVE Type Questions

1. The angular velocity of a wheel increases from 100 rps to 300 rps in 10 s. The number of revolutions made during that time is

(a) 600 (b) 1500

(c) 1000 (d) 2000

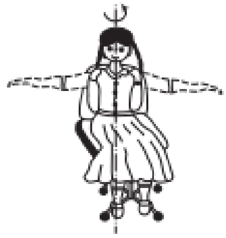
Sol. (d) Angular displacement θ during time t, assuming constant acceleration be

$$\theta = \frac{\omega_0 + \omega_t}{2}$$

$$= \frac{100 + 300}{2} \times 10$$

$$= 2000 \text{ revolutions}$$

2. If a girl rotating on a chair bends her hand as shown in figure the (neglecting frictional force).



(a) I_{girl} will reduce

(b) I_{girl} will increase

(c) ω_{girl} will reduce

(d) None of the above

Sol. (a) As there is no external torque, if the girl bends her hands, her MI about the rotational axis will decrease. By conservation of angular momentum,

if $L = I\omega$ = constant, then ω will increase.

3. When a disc rotates with uniform angular velocity, which of the following is not true? [NCERT Exemplar]

(a) The sense of rotation remains same

(b) The orientation of the axis of rotation remains same

(c) The speed of rotation is non-zero and remains same

(d) The angular acceleration is zero

Sol. (d) We know that angular acceleration

$$\alpha = \frac{d\omega}{dt}, \text{ given } \omega = \text{constant}$$

where ω is angular velocity of the disc

$$\Rightarrow \quad \alpha = \frac{d\omega}{dt} = \frac{0}{dt} = 0$$

Hence, angular acceleration is zero.

VERY SHORT ANSWER Type Questions

4. Why is moment of inertia also called rotational inertia?

Sol. The moment of inertia gives a measure of inertia in rotational motion.

So, it is also called rotational inertia.

5. Does the moment of inertia of a rigid body change with the speed of rotation?

Sol. No, because the moment of inertia depends upon the axis of rotation and distribution of mass.

6. Two solid spheres of the same mass are made of metals of different densities. Which of them has a larger moment of inertia about the diameter? Why?

Sol. The sphere with smaller density will have larger radius and hence larger moment of inertia.

7. In a flywheel, most of the mass is concentrated at the rim. Explain why?

Sol. Concentration of mass at the rim increases the moment of inertia and thereby brings uniform motion.

8. What is the moment of inertia of a sphere of mass 20 kg and radius $\frac{1}{4}$ m about its diameter?

Sol. Moment of inertia of a sphere about its diameter,

$$I = \frac{2}{5}mR^2 = \frac{2}{5} \times 20 \times \left(\frac{1}{4}\right)^2 = 0.5 \text{ kg-m}^2$$

9. Does the radius of gyration depend upon the speed of rotation of the body?

Sol. No, it depends only on the distribution of mass of the body.

10. About which axis would a uniform cube have a minimum rotational inertia?

Sol. In a uniform cube, the mass is more concentrated about a diagonal.

11. A constant torque of 120 N-m rotates its point of action by an angle of 30°. What is the work done by the torque?

Sol. Work done by the torque, $W = \tau\theta$

$$= 120 \times \left(\frac{30 \times \pi}{180}\right) = 20\pi \text{ J}$$

12. A ballet dancer stretches her hand out for slowing down. Name the conservation obeyed.

Sol. This is based on the conservation of angular momentum.

13. A wheel of moment of inertia 50 kg-m^2 about its own axis is revolving at a rate of 5 revolutions per second. What is its angular momentum?

Sol. Here, $I = 50 \text{kg-m}^2$, $\omega = 5$ rps $= 5 \times 2\pi$ rad/s

$\therefore$ Angular momentum $L = I\omega = 50 \times 10\pi = 500\pi$ J-s

14. Can the mass of body be taken to be concentrated at its centre of mass for the purpose of calculating its rotational inertia?

Sol. No, the moment of inertia greatly depends on the distribution of mass about the axis of rotation.

15. If a cube is melted and is casted into a sphere, does moment of inertia about an axis through centre of mass increases or decreases.

Sol. Moment of inertia of a sphere is less than that of a cube of same mass.

SHORT ANSWER Type Questions

16. A person is standing on a rotating table with metal spheres in his hands. If he withdraw his hands to his chest, what will be the effect on his angular velocity?

Sol. When the person withdraws his hands to his chest, his moment of inertia decreases. No external torque is acting on the system. So, to conserve angular momentum, the angular velocity increases.

17. Why does a solid sphere have smaller moment of inertia than a hollow cylinder of same mass and radius about an axis passing through their axis of symmetry? **[NCERT Exemplar]**

Sol. All mass of a hollow cylinder lies at a distance R from axis of rotation. Whereas in case of a sphere, most of mass lies at a distance less than R from axis of rotation. As moment of inertia is $\Sigma M_i R_i^2$, so sphere as a lower value of moment of inertia.

18. Two boys of the same weight sit at the opposite ends of a diameter of a rotating circular table. What happens to the speed of rotation if they move nearer to the axis of rotation?

Sol. The moment of inertia of the system (circular table + two boys) decreases. To conserve angular momentum ($L = I\omega =$ constant), the speed of rotation of the circular table increases.

19. If ice on poles melts, then what is the change in duration of day?

Sol. Molten ice from poles goes into ocean and so mass is going away from axis of rotation. So, moment of inertia of earth increases and to conserve angular momentum, angular velocity (ω) decreases. So, time period of rotation increases ($T = 2\pi/\omega$). So, net effect of global warming is increasing in the duration of day.

20. The speed of a whirl wind in a tornado is alarmingly high. Why?

Sol. In a whirl wind, the air from nearby region gets concentrated in a small space thereby decreasing the value of moment of inertia considerably. Since, $I\omega =$ constant, due to decrease in moment of inertia, the angular speed becomes quite high.

21. A solid cylinder of mass 20 kg rotates about its axis with angular speed of 100 rad/s. The radius of cylinder is 0.25 m. What is KE of rotation of cylinder? **[NCERT Exemplar]**

Sol. $M = 20$ kg, $\omega = 100$ rad/s, $R = 0.25$ m

Moment of inertia of cylinder about its own axis

$$= \frac{1}{2} MR^2 = \frac{1}{2} \times 20 \times (0.25)^2 = 0.625 \text{ kg-m}^2$$

Rotational KE $= \frac{1}{2} I\omega^2$

$$= \frac{1}{2} \times 0.625 \times (100)^2 = 3125 \text{ J}$$

22. If earth contracts to half its radius. What would be the length of the day?

Sol. The moment of inertia $\left(I = \frac{2}{5} MR^2\right)$ of the earth about its own axis will become one-fourth and so its angular velocity will become four times ($L = I\omega$ = constant). Hence, the time period will reduce to one-fourth ($T = 2\pi/\omega$), i.e. 6 hours.

23. Explain how a cat is able to land on its feet after a fall taking the advantage of principle of conservation of angular momentum?

Sol. When a cat falls to ground from a height, it stretches its body alongwith the tail so that its moment of inertia becomes high. Since, $I\omega$ is to remain constant, the value of angular speed ω decreases and therefore the cat is able to land on the ground gently.

LONG ANSWER Type I Questions

24. A rope of negligible mass is wound round a hollow cylinder of mass 3 kg and radius 40 cm. What is angular acceleration of the cylinder, if the rope is pulled with a force of 30 N? What is linear acceleration of the rope? Assume no slipping. **[NCERT]**

Sol. Torque on cylinder, τ = force × radius

$$= 30 \times 0.4 = 12 \text{ N-m}$$

Moment of inertia of hollow cylinder about its axis

$$I = MR^2 = 3 \times (0.4)^2 = 0.48 \text{ kg-m}^2$$

Also, $\tau = I\alpha \Rightarrow \alpha = \frac{\tau}{I}$

$\therefore$ $\alpha = \frac{12}{0.48} = 25 \text{ s}^{-2}$

Linear acceleration of rope

$$a = \frac{F}{m} = \frac{30}{3} = 10 \text{ m/s}^2$$

25. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having the same mass and radius. The cylinder is free to rotate about its standard axis of symmetry and the sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time? **[NCERT]**

Moment of inertia of the hollow cylinder about its axis of symmetry.

$$I_1 = MR^2$$

Moment of inertia of the solid sphere about its diameter, $I_2 = \frac{2}{5} MR^2$

Sol. Let M and R be the mass and radius of the solid sphere and hollow cylinder.

Let torque τ of equal magnitude be applied on hollow cylinder and solid sphere. The angular accelerations produced in it are α_1 and α_2, respectively.

$\therefore$ $\tau = I_1\alpha_1$ and $\tau = I_2\alpha_2$

Therefore, $I_1\alpha_1 = I_2\alpha_2$

or $$\frac{\alpha_1}{\alpha_2} = \frac{I_2}{I_1} = \frac{\frac{2}{5}MR^2}{MR^2} = \frac{2}{5}$$

or $$\alpha_2 = \frac{5}{2}\alpha_1 = 2.5\alpha_1 \quad \text{...(i)}$$

Let after time t, ω_1 and ω_2 be the angular speeds of the hollow cylinder and solid sphere, respectively.

$\therefore$ $$\omega_1 = \omega_0 + \alpha_1 t \quad \text{...(ii)}$$

and $$\omega_2 = \omega_0 + \alpha_2 t = \omega_0 + 2.5\,\alpha_1 t \quad \text{...(iii)}$$

From Eqs. (ii) and (iii), we get $\omega_2 > \omega_1$

Therefore, solid sphere will acquire a greater angular speed after a given time.

26. A wheel in uniform motion about an axis passing through its centre and perpendicular to its plane is considered to be in mechanical (translational and rotational) equilibrium because no net external torque is required to sustain its motion.

However, the particles that constitute the wheel to experience a centripetal acceleration towards the centre. How do you reconcile this fact with the wheel begin in equilibrium?

How would you set a half wheel into uniform motion about an axis passing through the centre of mass of the wheel and perpendicular to its plane? Will you require external forces to sustain the motion? **[NCERT Exemplar]**

Sol. The centripetal acceleration in a wheel arise due to the internal elastic forces which in pairs cancel each other. So the system remains in equilibrium during rolling.

In a half wheel, the system is not symmetrical so, direction of angular momentum does not coincide with the direction of angular velocity and hence an external torque is required to maintain the rotation.

27. A solid cylinder of mass 20 kg rotates about its axis with angular speed (100 rad/s). The radius of cylinder is 0.25 m. What is the kinetic energy associated with the rotation of the cylinder? What is the magnitude of angular momentum of the cylinder about its axis? **[NCERT]**

Sol. Moment of inertia of cylinder about its axis $= \frac{1}{2} MR^2$

M = mass, R = radius

$$= \frac{1}{2} \times 20 \times (0.25)^2 \text{ kg-m}^2$$

$$= 0.625 \text{ kg-m}^2$$

Kinetic energy of rotating cylinder

$$= \frac{1}{2} I\omega^2 = \frac{1}{2} (0.625)(100)^2 \text{ J}$$

$$= 3125 \text{ J}$$

Angular momentum of cylinder about its own axis

$$= I\omega = 0.625 \times 100$$

$$= 62.5 \text{ kg-m}^2/\text{s}$$

28. A hoop of radius 2 m weighs 100 kg. It rolls along a horizontal floor so that its centre of mass has a speed of 20 cm/s. How much work has to be done to stop it? **[NCERT Exemplar]**

Sol. Moment of inertia of hoop about its centre

$$I = MR^2$$

and energy of loop = translational kinetic energy of CM and rotational kinetic energy about axis through CM.

$$= \frac{1}{2} mv_{CM}^2 + \frac{1}{2} I\omega^2 \qquad \left[I = mR^2 \text{ and } \omega = \frac{v}{R}\right]$$

$$= \frac{1}{2} mv_{CM}^2 + \frac{1}{2} mR^2 \times \frac{v_{CM}^2}{R^2} = mv_{CM}^2$$

Work done is stopping the hoop

$$= \text{total KE of loop}$$

$$= mv_{CM}^2 = 100 \times (0.2)^2 = 4 \text{ J}$$

29. A bullet of mass 10 g and speed 500 m/s is fired into a door and gets embedded exactly at the centre of the door. The door is 1.0 m wide and weight 12 kg. It is hinged at one end and rotates about a vertical axis practically without friction. Find the angular speed of the door just after the bullet embeds into it. **[NCERT]**

Sol. Given, mass of bullet $(m) = 10 \text{ g} = 0.01 \text{ kg}$

Speed of bullet $(v) = 500$ m/s

Width of the door $(l) = 1.0$ m

Mass of the door $(M) = 12$ kg

As bullet gets embedded exactly at the centre of the door, therefore its distance from the hinged end of the door,

$$r = \frac{l}{2} = \frac{1}{2} \text{m}$$

Angular momentum transferred by the bullet to the door,

$$L = mv \times r = 0.01 \times 500 \times \frac{1}{2} = 2.5 \text{ Js}$$

Moment of inertia of the door about the vertical axis at one of its end, $I = \frac{Ml^2}{3} = \frac{12 \times (1)^2}{3} = 4 \text{ kg-m}^2$

But angular momentum, $L = I\omega$

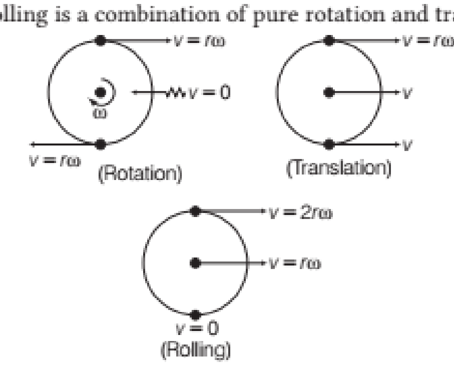
$$2.5 = 4 \times \omega = \frac{2.5}{4} = 0.625 \text{ rad/s}$$

30. A uniform circular disc of radius R is rolling on a horizontal surface. Determine the tangential velocity at

(i) upper most point (ii) at centre of mass

(iii) at point of contact

Sol. Rolling is a combination of pure rotation and translation



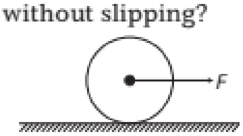
So, tangential velocity

(i) of uppermost point is $2r\omega$.

(ii) of centre of mass is $r\omega$.

(iii) of point of contact is zero.

31. A uniform disc of radius R is resting on a table on its rim. The coefficient of friction between disc and table is μ (figure). Now, the disc is pulled with a force F as shown in the figure. What is the maximum value of F for which the disc rolls without slipping?



[NCERT Exemplar]

Sol. Let the acceleration of the centre of mass of disc be a, then

$$Ma = F - f$$

The angular acceleration of the disc is $\alpha = a/R$ (if there is no sliding).

Then, $\left(\frac{1}{2} MR^2\right)\alpha = Rf \Rightarrow Ma = 2f$

Thus, $f = F/3$. Since, there is no sliding.

$$\Rightarrow \qquad f \le \mu mg \Rightarrow F \le 3\mu Mg$$

32. A cylinder of mass 10 kg and radius 15 cm is rolling perfectly on a plane of inclination 30°. The coefficient of static friction, $\mu_s = 0.25$.

(i) How much is the force of friction acting on the cylinder?

(ii) What is the work done against friction during rolling?

(iii) If the inclination θ of the plane is increased, at what value of θ does the cylinder begin to skid and not roll perfectly? **[NCERT]**

Sol. Given, mass of the cylinder, $m = 10$ kg

Radius, $r = 15$ cm $= 0.15$ m

Inclination of plane, $\theta = 30°$

Coefficient of static friction, $\mu_s = 0.25$

(i) Force of friction acting on the cylinder on the inclined plane, $F = \frac{1}{3} mg \sin\theta = \frac{1}{3} \times 10 \times 9.8 \times \sin 30°$

$$= \frac{1}{3} \times 10 \times 9.8 \times \frac{1}{2} = 16.3 \text{ N}$$

(ii) Force of friction acts perpendicular to the direction of displacement.

∴ Work done against friction during rolling

$$W = Fs \cos 90° = 0$$

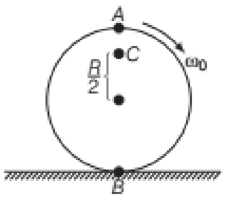
(iii) For rolling without slipping,

$$\mu = \frac{1}{3} \tan\theta$$

or $\tan\theta = 3\mu = 3 \times 0.25 = 0.75 = \tan 36°54'$

or $\theta = 36°54' = 37°$

33. Explain why friction is necessary to make the disc in figure rolling in the direction indicated.



(i) Give the direction of frictional force at B and the sense of frictional torque before perfect rolling begins.

(ii) What is the force of friction after perfect rolling begins? **[NCERT]**

Sol. To roll a disc, one require a linear velocity which can be provided only by a tangential force. As frictional force is the only tangential force in this case, so it is necessary for the rolling of the disc. Initially, friction will be kinetic.

(i) As frictional force at B opposes the angular velocity of B. So, frictional force is in the forward direction, the sense of frictional torque is such as to oppose the angular motion and produce some linear motion, so that the condition of pure rolling ($v_{CM} = R\omega$) should be fulfilled.

(ii) After pure rolling starts there will be no need of friction, so friction force will become zero.

LONG ANSWER Type II Questions

34. A solid disc and a ring, both of radius 10 cm are placed on a horizontal table simultaneously, with initial angular speed equal to 10π rads^{-1}. Which of the two will start to roll earlier? The coefficient of kinetic friction is $\mu_k = 0.2$? **[NCERT]**

In case of pure rotation without translation, the velocity of centre of mass is zero. The friction reduces the speed at the point of contact and as such accelerates the centre of mass till the velocity of centre of mass becomes equal to $v = R\omega$ and the instantaneous velocity at the contact point becomes zero.

Sol. Thus, the force of friction $\mu_k\, mg$ produces an acceleration a in the centre of mass. So, the equation of motion for centre of mass is

$$\mu_k mg = ma \quad \text{...(i)}$$

The torque due to force of friction is $\mu_k\, mg \times R$. It produces angular retardation given by

$$\mu_k\, mgR = -I\alpha \quad \text{...(ii)}$$

Rolling begins when $v = R\omega$

But $v = 0 + at = \mu_k gt$ ...(iii)

[from Eq. (i), $a = \mu_k g$]

and $\omega = \omega_0 + \alpha t = \omega_0 - \frac{\mu_k mgR}{I} t$ [using Eq. (ii)]

or $\frac{v}{R} = \omega_0 - \frac{\mu_k mgR}{I} t \Rightarrow \frac{\mu_k gt}{R} = \omega_0 - \frac{\mu_0 mgRt}{I}$

or $\frac{\mu_k gt}{R}\left[1 + \frac{mR^2}{I}\right] = \omega_0$ or $t = \frac{R\omega_0}{\mu_k g\left(1 + \frac{mR^2}{I}\right)}$

For a disc, $I = mR^2/2$

∴ $$t = \frac{R\omega_0}{3\mu_k g} = \frac{0.10 \times 10\pi}{3 \times 0.2 \times 9.8} = 0.53 \text{ s}$$

For a ring, $I = mR^2$

∴ $$t = \frac{R\omega_0}{2\mu_k g} = \frac{0.10 \times 10\pi}{2 \times 0.2 \times 9.8} = 0.80 \text{ s}$$

Thus, the disc begins to roll earlier than the ring.

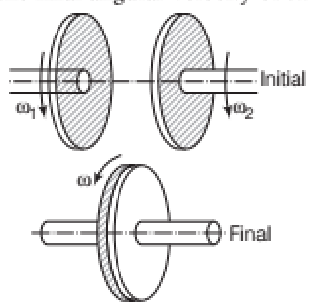
35. Two discs of moment of inertia I_1 and I_2 (about their axes normal to plane of disc and passing through centre) are rotating with angular speeds ω_1 and ω_2 are brought into contact face to face with their axes coinciding.

(i) What is angular speed of two discs system?

(ii) Show that the kinetic energy of combined system is less than the sum of initial kinetic energies of the two discs.

(iii) How do you account for loss of energy? **[NCERT]**

Sol. Let ω be the final angular velocity of two discs system.



(i) According to conservation of angular momentum, we get

Total initial angular momentum = Final angular momentum

$$\Rightarrow \quad I_1\omega_1 + I_2\omega_2 = (I_1 + I_2)\,\omega$$

$$\Rightarrow \quad \omega = \left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)$$

(ii) Initial KE of disc $= K_1 + K_2 = \frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2$

Final KE of disc system $= \frac{1}{2}(I_1 + I_2)\,\omega^2$

$$= \frac{1}{2}(I_1 + I_2)\cdot\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2$$

Loss of KE = Initial KE − Final KE

$$= \left(\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2\right) - \frac{1}{2}(I_1 + I_2)\cdot\left(\frac{I_1\omega_1 + I_2\omega_2}{I_1 + I_2}\right)^2$$

$$= \frac{1}{2(I_1 + I_2)}(I_1^2\omega_1^2 + I_1I_2\omega_2^2 + I_1I_2\omega_1^2 + I_2^2\omega_2^2 - I_1^2\omega_1^2 - I_2^2\omega_2^2 - 2I_1I_2\omega_1\omega_2)$$

$$= \frac{1}{2(I_1 + I_2)}(I_1I_2\omega_2^2 + I_1I_2\omega_1^2 - 2I_1I_2\omega_1\omega_2)$$

$$= \frac{1}{2(I_1 + I_2)}\cdot I_1I_2\cdot(\omega_2^2 + \omega_1^2 - 2\omega_1\omega_2)$$

$$= \frac{I_1I_2}{2(I_1 + I_2)}\cdot(\omega_1 - \omega_2)^2$$

As above quantity is positive, so a loss of energy occurs.

(iii) When 2 discs come in contact, they rub against each other till both reach same energy.

36. A man stands on a rotating platform with his arms stretched horizontally holding a 5 kg weight in each hand. The angular speed of the platform is 30 rpm. The man then brings his arms close to his body with the distance of each weight from the axis changing from 90 cm to 20 cm. The moment of inertia of the man together with the platform may be taken to be constant and equal to 7.6 kg-m^2.

(i) What is his new angular speed? (Neglect friction)

(ii) Is kinetic energy conserved in the process? If not, from where does the change come about? **[NCERT]**

Sol. (i) Moment of inertia of man and platform system

$I_i = 7.6$ kg-m^2

Change in moment of inertia of man and platform system when he stretches his hands to a distance of 90 cm $= 2 \times mr^2 = 2 \times 5 \times (0.9)^2$

$= 8.1$ kg-m^2

$I_i = I + 8.1 = 7.6 + 8.1 = 15.7$ kg-m^2

Initial angular velocity, $\omega_i = 30$ rpm

Initial angular momentum of system,

$L_i = I_i\omega_i = 15.7$ kg-m$^2 \times 30$ rpm

When man folds his hands to a distance of 20 cm,

Moment of inertia of man $= 2 \times mr^2 = 2 \times 5 \times (0.2)^2$

$= 0.4$ kg-m^2

So, final moment of inertia of man and platform system

$= 7.6 + 0.4 = 8$ kg-m^2

Final angular momentum of system

$L_f = I_f\omega_f = 8 \times \omega_f$

Equating initial and final values

$L_i = L_f$

$$\Rightarrow \quad \omega_f = \frac{15.7 \times 30}{8}$$

$= 58.88$ rpm

(ii) KE is not conserved in process.

$K_{\text{final}} > K_{\text{initial}}$

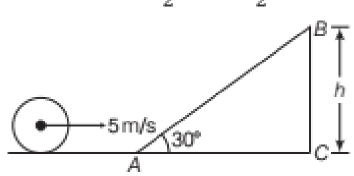
Muscular work done by the man in folding his arms is converted into KE.

37. A solid cylinder rolls up an inclined plane of angle of inclination 30°. At the bottom of inclined plane, the CM of cylinder has a speed of 5 m/s.

(i) How far will the cylinder go up the plane? **[NCERT]**

(ii) How long it will take to return to the bottom?

Sol. (i) Energy at point $A = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$



where, $I = \frac{1}{2}mr^2$ and $\omega = \frac{v}{r}$

So, energy at point $A = \frac{1}{2}mv^2 + \frac{1}{2} \times \frac{1}{2}mr^2 \times \frac{v^2}{r^2}$

$$= \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2$$

and energy at point $B = mgh$

Equating we get, $\frac{3}{4}mv^2 = mgh$

or $$h = \frac{3}{4}\frac{v^2}{g} = \frac{3}{4} \times \frac{5^2}{9.8} = 1.91 \text{ m}$$

(ii) In ΔABC, $$AB = \frac{h}{\sin 30^\circ}$$

or $$AB = \frac{1.91}{0.5} = 3.82 \text{ m}$$

So, cylinder rolls a direction of 3.82 m up the incline.

38. A disc of radius R is rotating with an angular speed ω, about a horizontal axis. It is placed on a horizontal table. The coefficient of kinetic friction is μ_k.

(i) What was the velocity of its centre of mass before being brought in contact with the table?

(ii) What happens to the linear velocity of a point on its rim when placed in contact with the table?

(iii) What happens to the linear speed of the centre of mass when disc is placed in contact with the table?

(iv) Which force is responsible for the effects in (ii) and (iii)?

(v) What condition should be satisfied for rolling to begin?

(vi) Calculate the time taken for the rolling to begin. **[NCERT Exemplar]**

Sol. (i) Before the disc is brought in contact with table, it is only rotating about its horizontal axis. So, its centre of mass is at rest, i.e. $v_{CM} = 0$.

(ii) When rim is placed in contact with the table, then its linear velocity at any point on the rim of disc will reduce due to kinetic friction.

(iii) If rotating disc is placed in contact with the table, its centre of mass acquires some velocity (which was zero before contact) due to kinetic friction. So, linear velocity of CM will increase.

(iv) Kinetic friction.

(v) Rolling will begin when $v_{CM} = R\omega$.

(vi) Acceleration produced in centre of mass due to friction $$a_{CM} = \frac{F}{m} = \frac{\mu_k mg}{m} = \mu_k g$$

Angular acceleration produced by the torque due to friction. $$\alpha = \frac{\tau}{I} = \frac{\mu_k mgR}{I}$$

$\therefore$ $$v_{CM} = u_{CM} + a_{CM}t$$

$\Rightarrow$ $$v_{CM} = \mu_k gt$$

and $$\omega = \omega_0 + \alpha t \Rightarrow \omega = \omega_0 - \frac{\mu_k mgR}{I}t$$

For rolling without slipping,

$$\frac{v_{CM}}{R} = \omega_0 - \frac{\mu_k mgR}{I}t$$

$\Rightarrow$ $$\frac{\mu_k gt}{R} = \omega_0 - \frac{\mu_k mgR_t}{I}$$

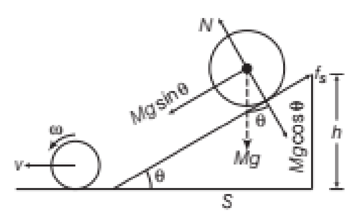
$$t = \frac{R\omega_0}{\mu_k g\left(1 + \frac{mR^2}{I}\right)}$$

39. Prove the result that the velocity v of translation of a rolling body (like a ring, disc, cylinder or sphere) at the bottom of an inclined plane of a height h is given by $v^2 = \frac{2gh}{(1 + K^2/R^2)}$ using dynamical consideration (i.e. by consideration of forces and torques). Note K is the radius of gyration of the body about its symmetry axis and R is the radius of the body. The body starts from rest at the top of the plane. **[NCERT]**

Sol. Let a body of mass M and radius R is rolling down a plane inclined at an angle θ with the horizontal and forces acting on the body are also shown in figure.

Let a be the downward acceleration of the body. The equations of motion for the body can be written as

$$N - Mg\cos\theta = 0$$
$$F = Ma = Mg\sin\theta - f$$



As the force of friction f provides the necessary torque for rolling, so

$$\tau = f \times R = I\alpha = MK^2\left(\frac{a}{R}\right)$$

or $$f = M\frac{K^2}{R^2} \cdot a$$

where, K is the radius of gyration of the body about its axis of rotation. Clearly,

$$Ma = Mg\sin\theta - M\frac{K^2}{R^2}\cdot a$$

or $$a = \frac{g\sin\theta}{(1 + K^2/R^2)}$$

Let h be height of the inclined plane and s the distance travelled by the body down the plane. The velocity v attained by the body at the bottom of the inclined plane can be obtained as follows

$$v^2 - u^2 = 2as$$

or $$v^2 - 0^2 = 2\cdot\frac{g\sin\theta}{(1 + K^2/R^2)}\cdot s$$

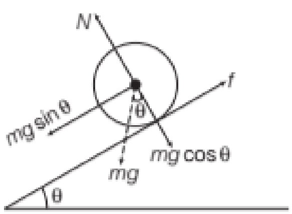
or $$v^2 = \frac{2gh}{1 + K^2/R^2} \quad \left[\because \frac{h}{s} = \sin\theta\right]$$

or $$v = \sqrt{\frac{2gh}{(1 + K^2/R^2)}}$$

40. **Obtain an expression for linear acceleration of a cylinder rolling down an inclined plane and hence find the condition for the cylinder to roll down the inclined plane without slipping.**

Sol. When a cylinder rolls down on an inclined plane, then forces involved are

(i) Weight mg (ii) Normal reaction N (iii) Friction f



From free body diagram,

$$N - mg\cos\theta = 0$$

or $$N = mg\cos\theta$$

Also, if a = acceleration of centre of mass down the plane, then

$$F_{net} = ma = mg\sin\theta - f \quad \text{...(i)}$$

As friction produces torque necessary for rotation,

$$\tau = I\alpha = fR$$

$$\Rightarrow \quad f = \frac{I\alpha}{R} = \frac{Ia}{R^2} \quad \left[\therefore \alpha = \frac{a}{R}\right]$$

Substituting for f in Eq. (i), we get

$$ma = mg\sin\theta - \frac{Ia}{R^2}$$

$$\Rightarrow \quad a = g\sin\theta - \frac{Ia}{mR^2}$$

For cylinder, $I = \frac{1}{2}mR^2$

$$\therefore \quad a = g\sin\theta - \frac{a}{2}$$

$$\Rightarrow \quad \frac{3a}{2} = g\sin\theta$$

$$\Rightarrow \quad a = \frac{2g\sin\theta}{3}$$

and from Eq. (i), the value of friction is

$$f = mg\sin\theta - ma$$

$$= mg\sin\theta - \frac{2}{3}mg\sin\theta$$

$$= \frac{1}{3}mg\sin\theta$$

If μ_s = coefficient of static friction, then

$$\mu_s = \frac{f}{N} \quad \text{or} \quad \mu_s = \frac{1}{3}\tan\theta$$

$\therefore$ For perfect rolling, $\mu_s \geq \frac{1}{3}\tan\theta$

ASSESS YOUR TOPICAL UNDERSTANDING

OBJECTIVE Type Questions

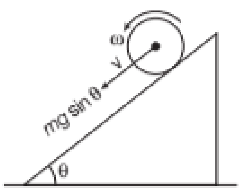
1. One solid sphere A and another hollow sphere B are of same mass and same outer radius. Their moments of inertia about their diameters are respectively, I_A and I_B such that

(a) $I_A = I_B$ (b) $I_A > I_B$
(c) $I_A < I_B$ (d) None of these

2. In pure rotation,

(a) all particles of the body move in a straight line
(b) all particles of body move in concentric circles
(c) all particles of body move in non-concentric circles
(d) all particles of body have same speed

3. Sphere is in pure accelerated rolling motion in the figure shown,



Choose the correct option.

(a) The direction of f_s is upwards
(b) The direction of f_s is downwards
(c) The direction of gravitational force is upwards
(d) The direction of normal reaction is downwards

4. A merry-go-round, made of a ring-like platform of radius R and mass M, is revolving with angular speed ω. A person of mass M is standing on it. At one instant, the person jumps off the round, radially away from the centre of the round (as seen from the round). The speed of the round of afterwards is **[NCERT Exemplar]**

(a) 2ω (b) ω
(c) $\frac{\omega}{2}$ (d) 0

Answer

1. (c)	2. (c)	3. (a)	4. (a)

VERY SHORT ANSWER Type Question

5. On which factors, moment of inertia of a rigid body depends?

SHORT ANSWER Type Question

6. A horizontal disc rotating about a vertical axis perpendicular to its plane and passing through centre makes 180 rpm. A small lump of wet mud of mass 10 g falls on disc lightly and sticks to it at a distance of 8 cm from its axis. If now the disc with mud makes 150 rpm only, calculate the moment of inertia of the disc.

[**Ans.** $I = 3.2 \times 10^{-8}$ kg-m^2]

7. Describe moment of inertia. Give two examples about how to find moment of inertia. Hence, describe radius of gyration.

8. Give three illustrations to show that moment of inertia is the rotational analogue of mass.

LONG ANSWER Type I Questions

9. Write three equations of motion for the rotational motion about a fixed axis.

10. Prove the relation $\tau = I\alpha$, for a rigid body rotating about a fixed axis.

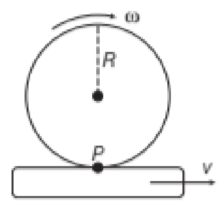
11. A disc is rolling on a horizontal floor with an angular speed ω and translation speed v. What fraction of total kinetic energy is rotational kinetic energy?

12. A diver bends his body while jumping in the swimming pool, due to this his angular speed increases. On which law, this incidence is based upon?

13. A body is in pure rolling on a horizontal road. What will be the net velocity of the uppermost point of that rolling body? The radius of rolling body is R and its angular velocity about axis passing through centre of mass is ω. [**Ans.** $2R\omega$]

LONG ANSWER Type II Questions

14. A sphere is rolling on a moving plank as shown in the above figure. What will be the velocity of centre of mass of the sphere? [**Ans.** $(v + R\omega)$]



15. A uniform disc of radius R and mass m is resting on table on its rim. The coefficient of friction between rim and table is μ. Now, disc is pulled with force F. What is the maximum value of F for which the disc rolls without slipping? [**Ans.** $F_{max} = 3\mu mg$]

16. A cylinder is released from rest from the top of an incline of inclination θ and length l. If the cylinder rolls without slipping. What will be its speed when it reaches the bottom? $\left[\textbf{Ans. } v = \sqrt{\frac{4}{3}gl\sin\theta}\right]$

SUMMARY

- A body is said to be a rigid body when it has a perfectly definite shape and size. The distance between all pairs of particles of such a body do not change while applying any force on it.
- The centre of mass of a body or a system of bodies is the point which moves as through all of the mass were concentrated there and all external forces were applied to it.
- For a system of n-particles, the centre of mass is given by $x_{CM} = \dfrac{m_1x_1 + m_2x_2 + \ldots + m_nx_n}{m_1 + m_2 + \ldots + mn} = \dfrac{\sum_{i=1}^{n} m_ix_i}{\Sigma m_i}$
- If in a two particles system, particles of masses m_1 and m_2 are moving with velocities v_1 and v_2 , respectively, then velocity of the centre of mass is given by $V_{CM} = \dfrac{m_1v_1 + m_2v_2}{m_1 + m_2}$
- If acceleration of the particles are a_1 and a_2 respectively, then acceleration of the centre of mass is given by $a_{CM} = \dfrac{m_1a_1 + m_2a_2}{m_1 + m_2}$
- The centre of mass of an object need not to lie within the object.
- The turning effect of a force about the axis of rotation is called moment of force or torque due to the force.
  Torque = Force × its perpendicular distance from the axis of rotation.
- The moment of linear momentum is called **angular momentum**. It is denoted by τ.
  Angular momentum (τ) = linear momentum $(p)\times$ its perpendicular distance from the axis of rotation (r).
- The rate of change of angular momentum of a system of particles about a fixed point is equal to the total external torque acting on the system about that point.
  $\tau_{total} = \dfrac{d}{dt}$
- **For translational equilibrium of a rigid body**, the vector sum of all the forces acting on a rigid body must be zero.

  i.e. $F_1 + F_2 + \ldots + F_n = \sum_{i=1}^{n} F_i = 0$
- **For rotational equilibrium**, the vector sum of torques of all the forces acting on the rigid body about the reference point must be zero.

  i.e. $\tau_1 + \tau_2 + \ldots + \tau_n = \sum_{i=1}^{n} \tau_i = 0$
- The moment of couple is equal to the product of either of the forces and the perpendicular distance, called the arm of the couple, between their lines of action.
  Moment of couple = Force × Arm of the couple
  or $\tau = Fd$
- In rotational equilibrium, Clockwise moment = Anti-clockwise moment
  or $F_1 \times d_1 = F_2 \times d_2$
  Or load × load arm = effort × effort arm
- The moment of inertia of a body about a given axis is equal to the sum of the products of the masses of its constituent particles and the square of their respective distances from the axis of rotation.
  Moment of inertia of a body is given by
  $I = m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots = \sum_{i=1}^{n} m_ir_i^2$
  Its unit is kg-m^2 and its dimensional formula is [ML^2].
  Moment of inertia, $I = 2 \times$ Rotational KE
- If $I = MK^2$
  Radius of gyration, $K = \sqrt{\dfrac{I}{M}}$

  For a body composed of particles of equal masses, $K = \sqrt{\dfrac{r_1^2 + r_2^2 + \ldots + r_4^2}{n}}$

  i.e. radius of gyration is equal to the root mean square distance of the particles from the axis of rotation.
- Angular momentum = Moment of inertia × Angular velocity i.e. $L = I_\omega$

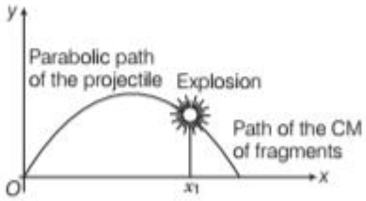
  or Torque = Moment of inertia × Angular acceleration, i.e. $L = I_\alpha$
- Rolling motion is very common in daily life. Motion of wheels of all modes of transportation is rolling motion. In fact, rolling motion is a combination of translation and rotation.

CHAPTER PRACTICE

OBJECTIVE Type Questions

1. The centre of mass of a system of two particles divides the distance between them.
   (a) In inverse ratio of square of masses of particles
   (b) In direct ratio of square of masses of particles
   (c) In inverse ratio of masses of particles
   (d) In direct ratio of masses of particles

2. A projectile is fired at an angle and it was following a parabolic path. Suddenly, it explodes into fragments. Choose the correct option regarding this situation.



   (a) Due to explosion CM shifts upwards
   (b) Due to explosion CM shifts downwards
   (c) Due to explosion CM traces its path back to origin
   (d) CM continues to move along same parabolic path

3. For rotational equilibrium,
   (a) $\sum_{i=1}^{n} \mathbf{F}_{\text{net}} = 0$ (b) $\sum_{i=1}^{n} \tau_{\text{net}} = 0$
   (c) Both (a) and (b) are the necessary conditions for the rotational equilibrium
   (d) Both (a) and (b) are not necessary for rotational equilibrium

4. When acrobat bends his body (assume no external torque)
   (a) I_{acrobat} decreases
   (b) I_{acrobat} increases
   (c) ω_{acrobat} increases
   (d) Both (a) and (c)

5. A drum of radius R and mass M rolls down without slipping along an inclined plane of angle θ. The frictional force
   (a) converts translational energy into rotational energy
   (b) dissipates energy as heat
   (c) decreases the rotational motion
   (d) decreases the rotational and translational motion

ASSERTION AND REASON

Direction (Q. Nos. 6-13) *In the following questions, two statements are given- one labelled Assertion (A) and the other labelled Reason (R). Select the correct answer to these questions from the codes (a), (b), (c) and (d) as given below*
   (a) Both Assertion and Reason are true and Reason is the correct explanation of Assertion.
   (b) Both Assertion and Reason are true but Reason is not the correct explanation of Assertion.
   (c) Assertion is true but Reason is false.
   (d) Assertion is false but Reason is true.

6. Assertion The CM of a body must lie on the body.
   Reason The CM of a body lie at the geometric centre of body.

7. **Assertion** The motion of the CM describes the translational part of the motion.
   Reason Translational motion always means straight line motion.

8. **Assertion** At the centre of earth, a body has centre of mass, but no centre of gravity.
   Reason Acceleration due to gravity is zero at the centre of earth.

9. **Assertion** Inertia and moment of inertia are same quantities.
   Reason Inertia represents the capacity of a body to oppose its state of motion of rest.

10. **Assertion** Moment of inertia of a particle is same, whatever be the axis of rotation.
    Reason Moment of inertia depends on mass and distance of the particle from the axis of rotation.

11. **Assertion** A particle moving on a straight line with a uniform velocity, its angular momentum is constant.
    Reason The angular momentum is zero when particle moves with a uniform velocity.

12. **Assertion** For a system of particles under central force field, the total angular momentum is conserved.
    Reason The torque acting on such a system is zero.

13. **Assertion** If bodies slide down an inclined plane without rolling, then all bodies reach the bottom simultaneously.

Reason Acceleration of all bodies are equal and independent of the shape.

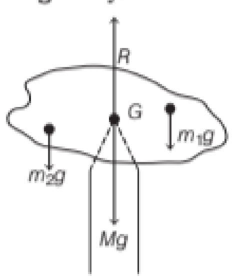
CASE BASED QUESTIONS

Directions (Q. Nos. 14-15) *These questions are case study based questions. Attempt any 4 sub-parts from each question.*

14. The Centre of Gravity

The centre of gravity of a rigid body is a point through which the total weight of the body act. Centre of gravity can lie within the body or not. For small objects, centre of mass always coincides with centre of gravity. But in case of large bodies whose dimensions are comparable to the size of earth, centre of mass and centre of gravity will be at different locations.

In the given figure, balancing of a cardboard on the tip of a pencil is done. The point of support, G is the centre of gravity.



(i) If the $\mathbf{F}_{\text{net, ext}}$ is zero on the cardboard, it means

(a) $R = Mg$ (b) $m_1 g = Mg$
(c) $m_2 g = Mg$ (d) $R = m_1 / g$

(ii) Choose the correct option.

(a) τ_{Mg} about CG $= 0$
(b) τ_R about CG $= 0$
(c) Net τ due to $m_1 g, m_2 g \cdots m_n g$ about CG $= 0$
(d) All of the above

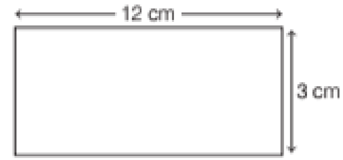
(iii) The centre of gravity and the centre of mass of a body coincide when

(a) g is negligible (b) g is variable
(c) g is constant (d) g is zero

(iv) If value of g varies, the centre of gravity and the centre of mass will

(a) coincide
(b) not coincide
(c) become same physical quantities
(d) None of the above

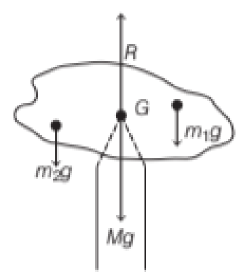
(v) Where will be the centre of gravity of the following rigid body?



(a) (6,3) (b) (6,6)
(c) (6,1.5) (d) (1.5,3)

15. Torque and Centre of Gravity

Torque is also known as moment of force or couple. When a force acts on a particle, the particle does not merely move in the direction of the force but it also turns about some point. So, we can define the torque for a particle about a point as the vector product of position vector of the point where the force acts and with the force itself. In the given figure, balancing of a cardboard on the tip of a pencil is done. The point of support, G is the centre of gravity.



(i) If the $\mathbf{F}_{\text{net, ext}}$ is zero on the cardboard, it means

(a) $R = Mg$ (b) $m_1 g = Mg$
(c) $m_2 g = Mg$ (d) $R = m_1 / g$

(ii) Choose the correct option.

(a) τ_{mg} about CG $= 0$
(b) τ_R about CG $= 0$
(c) Net τ due to $m_1 g, m_2 g \cdots m_n g$ about CG $= 0$
(d) All of the above

(iii) The centre of gravity and the centre of mass of a body coincide when

(a) g is negligible (b) g is variable
(c) g is constant (d) g is zero

(iv) If value of g varies, the centre of gravity and the centre of mass will

(a) coincide
(b) not coincide
(c) become same physical quantities
(d) None of the above

(v) A body lying in a gravitational field is in stable equilibrium, if
(a) vertical line through CG passes from top
(b) horizontal line through CG passes from top
(c) vertical line through CG passes from base
(d) horizontal line through CG passes from base

Answer

1. (c)	2. (c)	3. (b)	4. (d)	5. (a)
6. (d)	7. (c)	8. (a)	9. (d)	10. (d)
11. (c)	12. (a)	13. (c)		
14. (i) (a)	(ii) (d)	(iii) (c)	(iv) (b)	(v) (c)
15. (i) (a)	(ii) (d)	(iii) (c)	(iv) (b)	(v) (c)

VERY SHORT ANSWER Type Questions

16. What is the significance of centre of mass?

17. Three point masses of 1 kg, 2 kg and 3 kg lie at (1, 2), (0, −1) and (2, − 3), respectively. Calculate the coordinates of the centre of mass of the system. [Ans. 7/6, − 3/2]

18. What is the condition for precession?

19. Find the value of torque if $\mathbf{F} = (4\hat{\mathbf{i}} - 10\hat{\mathbf{j}})$ N and $\mathbf{r} = (-5\hat{\mathbf{i}} - 3\hat{\mathbf{j}})$ m [Ans. $62\,\hat{\mathbf{k}}$ N-m]

20. What is couple?

21. An automobile engine develops 100 kW when rotating at a speed of 1800 rpm. Find the torque produced. [Ans. 531 N-m]

22. State conservation of angular momentum theorem for a rigid body.

23. How to define rolling motion?

24. If the value of acceleration due to gravity (g) varies, will the centre of mass and centre of gravity coincide?

25. Write one example of conservation of angular momentum theorem.

SHORT ANSWER Type Questions

26. If two point masses are placed at $(+2\,m)$, and $(-2\,m)$, is it necessary that the centre of mass of system must lie at origin? [Ans. No]

27. Write two differences between centre of mass and centre of gravity.

28. State Newton's second law for rotational motion of a rigid body about a fixed axis.

29. A fan of moment of inertia is 0.6 kg-m^2 is to be run upto a working speed of 0.5 rps. What is the angular momentum of the fan? [Ans 1.9 kg-m/s]

LONG ANSWER Type I Questions

30. Moment of inertia of a thin rod of length l about an axis passing through its one end and perpendicular to its length is $MR^2/3$. Find the value of radius of gyration for the given scenario. [Ans. $K = F/\sqrt{3}$]

31. How to find kinetic energy of a rolling body?

32. A lay falls from the first floor of a building. He saw a pile of wool on the ground. He has a bag in his hand. Can he save himself with the help of that bag?

33. Derive the relation between angular momentum and moment of inertia for a rigid body rotating about a fixed axis.

34. Write three practical applications of conservation of angular momentum for a rigid body.

LONG ANSWER Type II Questions

35. Derive expressions for (i) Linear momentum, (ii) Newton's second law and (iii) Conservation of linear momentum for the system of particles.

36. Give location of centre of mass of (i) a sphere, (ii) cylinder, (iii) ring and (iv) a cube, each of uniform mass density. Does the centre of mass of a body necessarily lie inside the body?

37. Torques of equal magnitude are applied to a hollow cylinder and a solid sphere, both having same masses and radii. The cylinder is free to rotate about its standard axis of symmetry and sphere is free to rotate about an axis passing through its centre. Which of the two will acquire a greater angular speed after a given time?